Victorian Essential Learning Standards

CAS Technology Mathematics at Level 6

Part 5: Case Studies – CAS in Practice

Contents

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41 St Andrews Place East Melbourne Victoria 3002 Australia

TELEPHONE +61 3 9651 4300 FACSIMILE +61 3 9651 4324

EMAIL vcaa@edumail.vic.gov.au

WEB www.vcaa.vic.edu.au

Victoria The Place To Be

Case studies – using CAS in three schools

Following, is one of three sets of course outline materials and related activities developed and used by teachers who have worked with CAS as part of the Victorian Curriculum and Assessment Authority's (VCAA) Mathematical Methods (CAS) pilot program. In this case study, a teacher who worked in a CAS pilot school from 2001 to 2004 subsequently moved to a different school preparing for broader implementation of CAS enabled mathematics curriculum. In particular, this involved:

- a change in CAS platform, from the parallel use of *Derive* and TI-92+ to the use of TI-*n*spire CAS
- teacher professional learning through the Universities of Ballarat and Melbourne RITE Maths project and the New Zealand Qualifications Authority CAS pilot, resulting in the development of new learning activities and increasing teacher use of CAS as an enabling technology through curriculum refinement
- a change of schools, resulting in a shift in focus of pedagogy and content applicable to middle school mathematics curriculum, in particular, as possible preparation for subsequent study of Mathematical Methods (CAS) Units 1 and 2
- attention to the balance and interplay between content and process.

Case study C

School
Elisabeth Murdoch Secondary College – government, co-educational, 7–12
Curriculum structure
Elisabeth Murdoch College provides access to technology through spreadsheets and the TI-83 family of graphics calculators. In September 2005, the school introduced TI-92+ calculators in the junior and middle school. In 2006, the school became a Texas Instruments pilot school for the TI-*n*spire hand-held CAS. From 2006 and 2007, middle and junior school classes used these CAS, as well as students studying General Mathematics Units 1 and 2. Thus, some students studied General Mathematics Units 1 and 2 (CAS enabled) in conjunction with Mathematical Methods Units 1 and 2 (graphics calculator enabled). That is, the use of CAS was principally as a pedagogical tool to enhance student conceptual understanding. At Elisabeth Murdoch College mathematics is timetabled for 70-minute lessons, three times a week in Years 7 to 9 and four times a week in Year 10. This required a distinctive learning environment, with more material and concepts to be covered in a single lesson. Students in the middle and junior schools did not have individual ownership of the relevant enabling technology. This shared access requires greater scaffolding with respect to establishing familiarity with calculator functionality and the corresponding navigation, than when students have their own calculator.

CAS technology used
Elisabeth Murdoch Secondary College: TI-92+, TI-*n*spire CAS

Broad topic block planner

Year 9			
Semester 1		**Semester 2**	
Weeks	**Topic**	**Weeks**	**Topic**
1–3	Number systems	1–3	Chance and data (statistics)
4–6	Indices	4–7	Linear graphs
7–9	Pythagoras' theorem	8–11	Chance and data (probability)
10–14	Measurement	12–14	Trigonometry
15–18	Algebra	15–17	Geometry

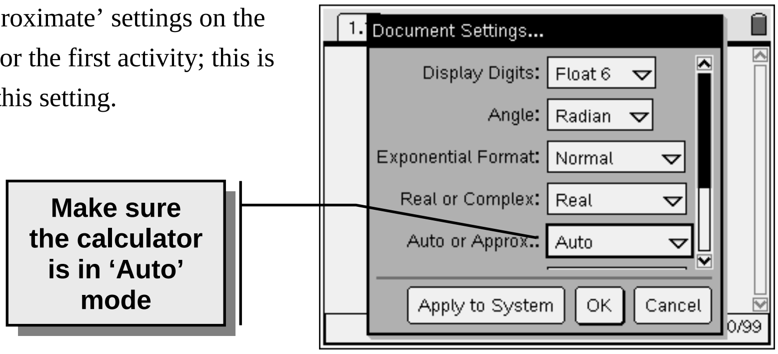
Increased student ownership of the mathematics they are learning is one of the significant changes to teaching and learning in a CAS enabled environment. This is particularly evident in topics such as surds. An important development in technology functionality from the earlier Texas Instruments TI-92+ (or TI-89) CAS to TI-*n*spire CAS has been incorporation of files called 'documents' as the natural working environment. Students open/save their work as a document, much the same as they would on a computer. These documents can be emailed to the teacher or used by the student to reflect upon their explorations, make conjectures and draw conclusions, which are all important aspects of working mathematically.

Initially, teachers might focus on the use of a CAS calculator for 'algebra' in a similar way to focusing on the use of graphics calculators for 'graphing'. However, these tools are far more versatile, and provide opportunities for the use of a combination of functionalities involving numerical (exact and approximate), graphical and symbolic computation, spreadsheets and dynamic geometry as the following examples illustrate. These examples include various teaching comments/suggestions.

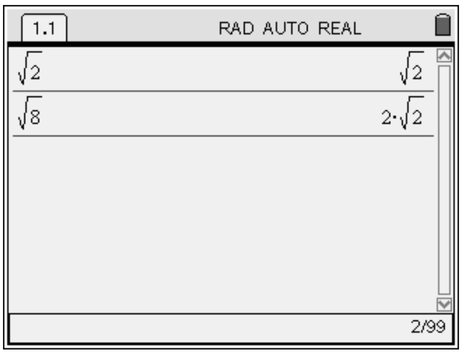
Year 9 program
Surds

Timeline Semester 1	Level 6 Year 9 Topic	VELS Learning focus	CAS implementation
Weeks 1–3	**Number systems** The real number system Surds operations +, - , ×, rationalise simple denominators	*Number* • set of real numbers • irrationals • surds • exact and approx context • arithmetic computations *Structure* • properties of real number systems • use of surds in calculations • form and test mathematical conjectures *Working mathematically* • use technology to assist mathematical inquiry	Introduction to surds Example 1 Operations with surds Example 2

If the exercises included here are to be used as a complete sequence, it is important that students are NOT told about the 'Auto / Exact / Approximate' settings on the calculator. Set the calculator in 'Auto' for the first activity; this is to ensure students are not conscious of this setting.



Example 1 – surds



Prior to commencing this exercise students already have an *understanding* of 'square root', and the related issue of sign. This *understanding* generally amounts to 'it's the opposite of squaring' and $\sqrt{\ }$ designates a non-negative number. Some formative assessment generally reveals student understanding of the real number system is limited to whole numbers, fractions and *decimal*.

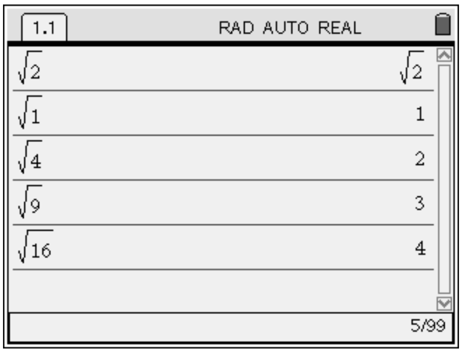
For this series of questions the calculator is set in *auto* mode which means that numbers such as $\sqrt{2}$ are not approximated. The calculator's automatic simplification of $\sqrt{8}$ suggests that the provision scaffolding to support student learning and the development of appropriate conjectures can amount to a meaningful selection of numbers (see following examples).

This exercise focuses on the simplification and manipulation of surds. It does not attempt to foster a conceptual understanding of surd irrational numbers.

Typical numbers for students to begin exploring include:

$\sqrt{1}$, $\sqrt{4}$, $\sqrt{9}$, $\sqrt{16}$ … $\sqrt{225}$

Students should recognise some of these numbers as perfect squares. It is therefore no surprise for the students that the results are evaluated. This is the same as the results generated by scientific and graphics calculators and confirms their understanding of square roots.



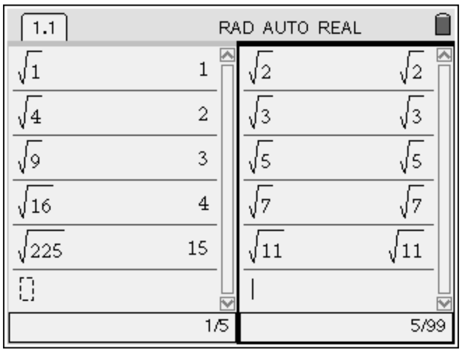
The next set of numbers for students to evaluate include:

$\sqrt{2}$ $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{11}$ … $\sqrt{23}$

These numbers are a set of square roots of prime numbers. The calculator does not approximate these values and simply returns the input expression. Students are required to comment and make conjectures about what is happening. Naturally, students suggest that 'these numbers are not evaluated because they are prime …'

At this point some students will automatically start investigating other numbers in order to check their conjecture.

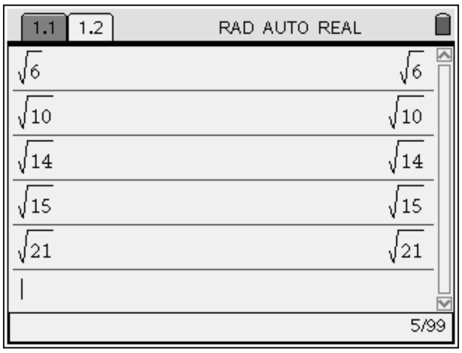
A split screen has been used here to display more *history* on a single screen. The calculator is capable of recalling 99 sets of calculations on a single page. However, it is often important for students to be able so see these values.



The next set of numbers provided to students to investigate include:

$\sqrt{6}$, $\sqrt{10}$, $\sqrt{14}$, $\sqrt{15}$, $\sqrt{21}$ …

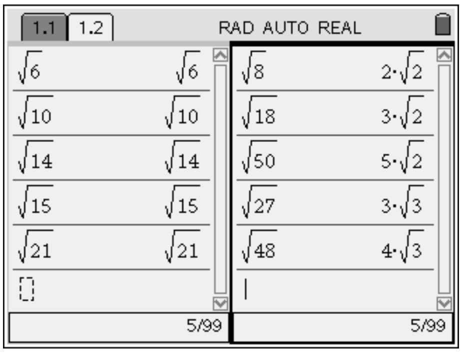
These numbers challenge student belief that it is only prime numbers that are not evaluated by the calculator. It is important to note that these numbers do not contain factors that are perfect squares. The conclusion students reach at this point amounts to 'the calculator only evaluates numbers that are perfect squares'.



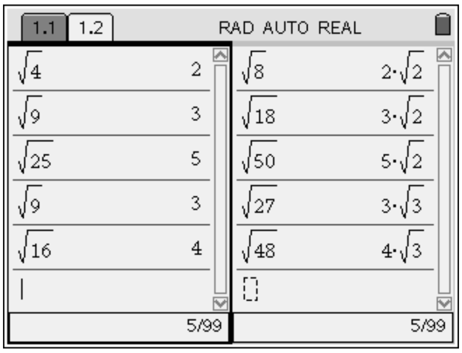
The next set of numbers provided to students to investigate include:

$\sqrt{8}$, $\sqrt{18}$, $\sqrt{50}$, $\sqrt{27}$, $\sqrt{48}$ …

Each one of these numbers has at least one factor that is a perfect square. The calculator automatically simplifies these expressions.



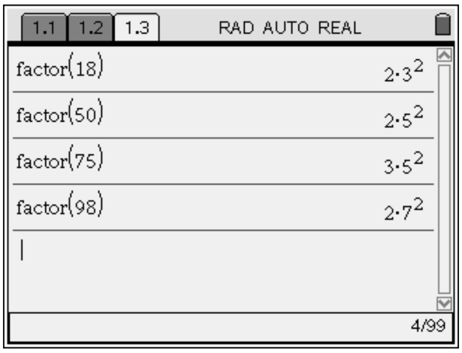
To provide additional support for some students, careful selection of the examples used across a split screen may help in their discovery of what the calculator is doing.



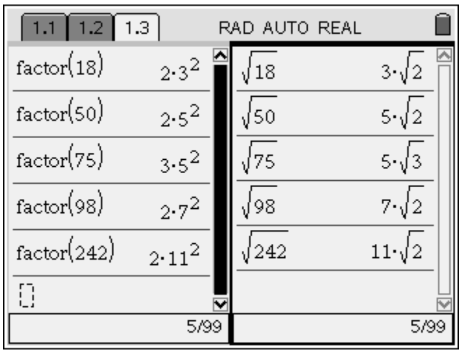
By this stage students are making conjectures 'when the number has a perfect square as one of its factors …'

The factor command on the calculator can be used to help students determine how the calculator may express numbers as a 'mixed' surd.

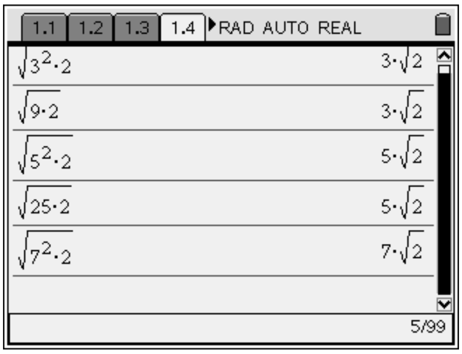
Again, careful selection of such numbers will provide sufficient scaffolding for students. For example: 18, 50, 75 … are numbers that contain *exactly* one perfect square. More complicated examples should be introduced gradually so that the exercise does not become overly complicated and involved with index laws.



Comparing the results of the factor command with how the calculator expresses a mixed surd is very powerful. Again, it is helpful to use the split screen to highlight this relationship.



Combining the results from the previous split screen, students can see another helpful representation. Note that the perfect square has been positioned at the start of the expression to further scaffold student learning.

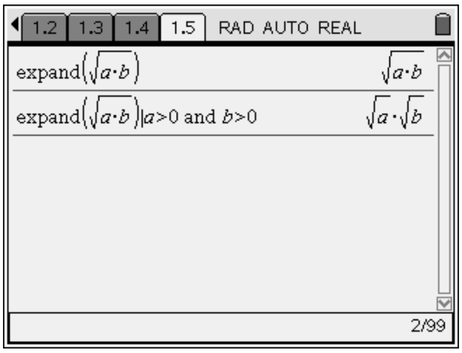


The previous exercises can lead to a ‘general’ solution for students, often included as:

$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$

This generalisation is often provided to students without qualification. Students need to understand that *a* and *b* must be the same sign; in these examples positive integers. The calculator also reinforces attention to such restrictions.

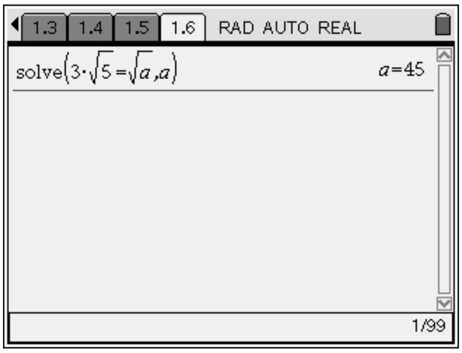
Attempting to ‘expand’ $\sqrt{a \times b}$ without appropriate domain restrictions does not change the expression. Once the domain restrictions are applied, the desired result is achieved.



Use of CAS calculators in middle school assessment tasks requires careful consideration. For example, students are asked to solve the equation:

$3\sqrt{5} = \sqrt{a}$

This can be done using CAS as shown, but students will also need to follow the mental computation, $3\sqrt{5} = \sqrt{9}\sqrt{5} = \sqrt{45}$, if they are to be able to recognise that the result is correct.

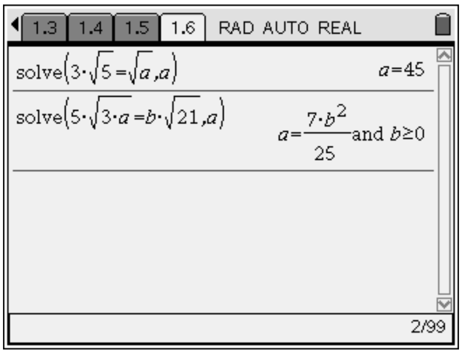


Consider the question:

Determine three sets of integer values for *a* and *b* for which:

$5\sqrt{3a} = b\sqrt{21}$

Using the solve command in this case produces an algebraic expression for *a* in terms of *b*, that is, a parametric solution. This result leads to students interpreting a set of possible solutions in terms of the original relation. Clearly, *a* = 0 and *b* = 0 is the trivial solution.



Consider the question once again:

Determine three sets integer values for *a* and *b* where: $5\sqrt{3a} = b\sqrt{21}$

Inspection also gives another solution *a* = 7 and *b* = 5

Students identify 3 × *a* = 21, therefore *a* = 7 and hence *b* = 5.

Another solution is *a* = 28 since 28 = 7 × 4 and therefore *b* = 10.

Students identify that perfect squares will be ‘removed’ from the square root; therefore, multiplying 7 by a perfect square will achieve the desired result.

Further solutions could include *a* = 63 since 63 = 7 × 9 and therefore *b* = 15. Alternatively, students could note that if $a = \frac{7b^2}{25}$ than *a* will be an integer when $7b^2$ is a multiple of 25. That is, when b^2 is a multiple of 25. This includes each of the previously identified solutions where *b* = 0, 5, 10 and 15 respectively.

The above example demonstrates how more general questions that require demonstration of a *conceptual knowledge* of simplification of surds can be posed in an environment where CAS calculators are available. These questions are more conceptually demanding than straightforward simplification and involve a general argument for the form of the solution.

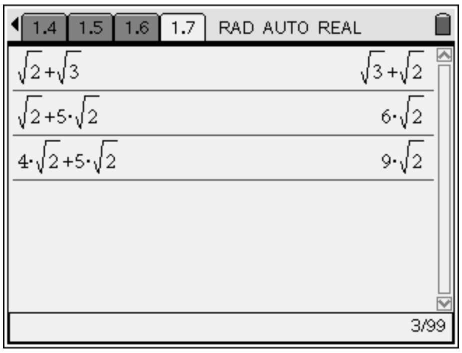
The investigation thus far aims to provide students with an insight as to *how* the calculator represents surds. It does not explain *why* the numbers have been represented in this way. It is now useful to address the number line representation of surds and elaborate how students have progressively learnt about different subsets of real numbers:

- Starting with 'counting numbers' (whole numbers) – lower primary
  Whole numbers represent a discrete model of the number line easily represented by counting physical objects.
- Fractions (introduction to rational numbers) – middle/upper primary
  The region between two whole numbers is divided into a discrete number of parts usually modelled by physically dividing objects into pieces. These parts fill in some of the gaps between the whole numbers along the number line. The number system is still discrete at this level; students are not conceptually ready to understand the continuity of the Real number system. The sequencing of concepts within VELS takes into consideration the various stages of students' abstract/concrete … thinking.
- Decimal – middle/upper primary
  Decimals provide a different representation of the fractions dealt with previously. However, conceptually they present a more complex situation to understand, as the 'division of the whole' is more abstract. Students practice converting between decimals and fractions and discover that some fractions such as 1/3 are not as easy to represent in decimal form. The number line is still discrete at the level as it is still only being represented by the set of rational numbers – a point often not understood by students.
- Negative numbers – upper primary/lower secondary
  An extension of the number system … below zero, but still discrete
- Irrational numbers – middle secondary
  Students will already be aware of specific irrational numbers (transcendental numbers) such as π through measurement. The unit on surds should address the realisation that infinitely many 'irrational' numbers exist. The exercises here demonstrate how the calculator deals with irrational numbers and provides some insight as to why the calculator handles them in a specific way. An understanding of irrational numbers needs to be developed further through a variety of classroom activities.

Example 2 – surds

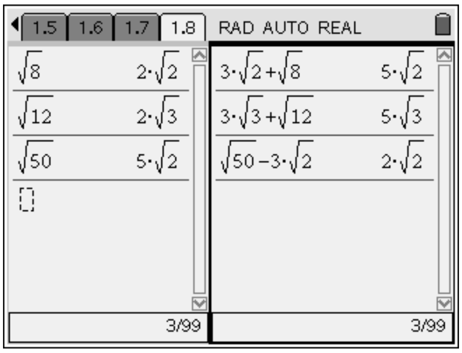
Students are required to *use surds in calculations*.

Using suitably selected examples, students can explore other operations involving surds using a similar approach to the previous example.

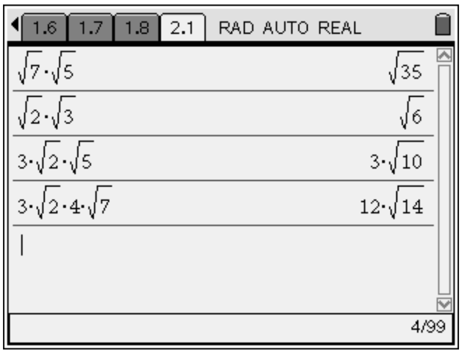


Slightly more complicated questions can be scaffolded for students using the split screen capabilities.

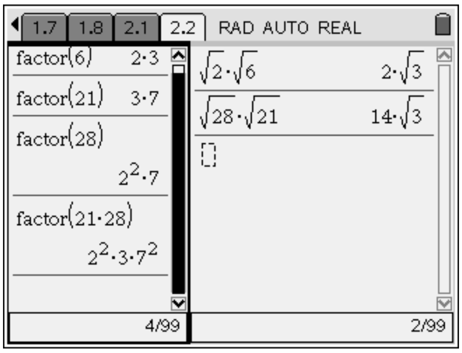
The screen on the left in this case acts to support students that do not immediately recognise some of the previous simplifications.



Other operations involving surds can also be explored.



As more complicated examples are selected, the split screen can once again be used to provide some scaffolding for less able students. This also relates to the product of powers of prime factors representation of natural numbers.

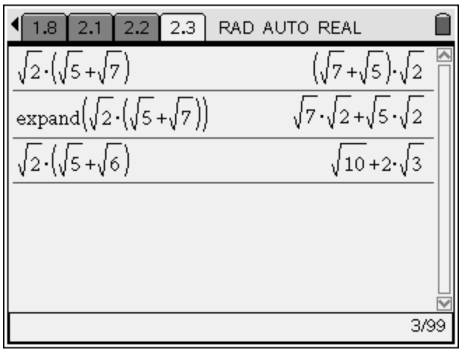


Prior to commencing a unit on surds that incorporates student exploration, teachers need to be aware of the variety of 'unexpected' outputs generated by the calculator. While teachers abide by specific rules, it is important that they are aware of how such expressions will be treated, particularly if the intention is to provide some scaffolding for student learning.

What would you expect from: $\sqrt{2}\left(\sqrt{5}+\sqrt{7}\right)$?

What about: *Expand* $\left(\sqrt{2}\left(\sqrt{5}+\sqrt{7}\right)\right)$?

How does this compare with: $\sqrt{2}\left(\sqrt{5}+\sqrt{6}\right)$?



Questions relating to operations with surds include:

determine at least one set of integer values for *a* and *b* given: $\sqrt{7}\left(\sqrt{63}+\sqrt{a}\right)=b$.

Such questions require students to understand that *a* must be a multiple of 7 in order for *b* to be rational.

One of the key features of this surds unit is that it arms students with a number of investigative tools:

- identifying patterns
- systematic exploration
- formulating AND testing conjectures
- familiarity with CAS technology.

These are important steps as a student progresses through VELS Level 6. In the Working mathematically dimension of VELS, specific reference is made to: '*Students* choose, use and develop mathematical models and procedures to investigate and solve problems set in a wide range of practical, theoretical …'One of the key factors for consideration provided here is that *students choose*. Students need to be working towards this goal and providing them with investigations such as this Surds unit assists them to prepare for independent learning through investigations and problem solving.

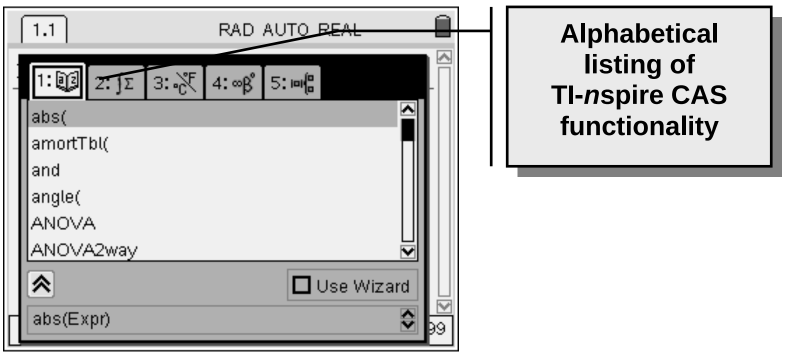
Year 9 program
Indices

Timeline Semester 1	Level 6 Year 9 Topic	VELS Learning focus	CAS implementation
Weeks 4–6	**Indices** • factor trees • index laws	*Number* • lowest common multiple *prime factors* • perform computations involving very large/small numbers *Structure* • form and test mathematical conjectures • simplify algebraic expressions • working mathematically • use technology to assist mathematical inquiry	LCMs Example 1 Index laws Example 2 Index laws Example 3

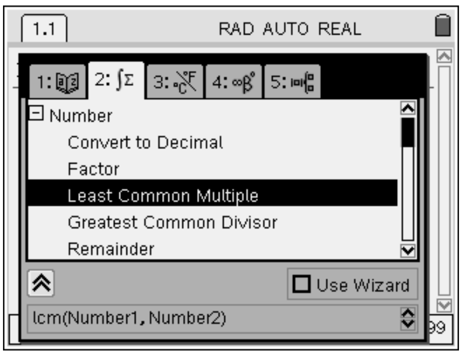
In Years 7 and 8 (VELS Level 5) students will have worked with factor trees as a means of identifying prime factors. Some students will have used CAS calculators to see how the **Factor** command represents these prime factors. In so doing, indices are introduced to students at Level 5.

In Year 9 (Level 6), these concepts are extended and used to help identify the lowest common multiple of a set of numbers. The limitations of the calculator prove very useful here as a learning tool. Concepts developed in this unit prove useful for factorisation units later in Level 6.

The TI-*n*spire CAS has two catalogues. One is an alphabetical listing of the calculator's functionality. This is useful if one knows the name of the function.



The catalogue also occurs in a mathematical listing where functionality is identified by its applicable mathematical context. In the LCM investigation, students need to find out if the calculator can determine the 'lowest common multiple' of a set of numbers. An appropriate category for such a tool would be 'Number'. The screen shown here lists some of the other functionality under this category. Other sub-categories exist within 'Number' such as fraction tools, which include such functionality as 'common denominator', 'Proper Fraction' and so forth. While these functions may be applicable to 'Number' for junior and middle school students, they hold algebraic significance for senior mathematics students extending to differential and integral calculus.



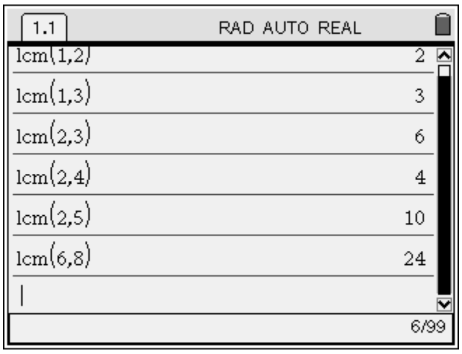
Here are a number of useful editing tips for students in this investigation:

- use the up/down navigation tool to highlight previously used expressions. Pressing ‘ENTER’ copies and pastes this command to the active line.
- if part of an expression is required, use the shift (⇧) key to highlight a selection of an expression.
- pressing CTRL – C is equivalent to Copy
- pressing CTRL – V is equivalent to Paste.
- when a command is known, it can be typed in directly using the keyboard: lcm (…).

Example 1 – lowest common multiple (lcm)

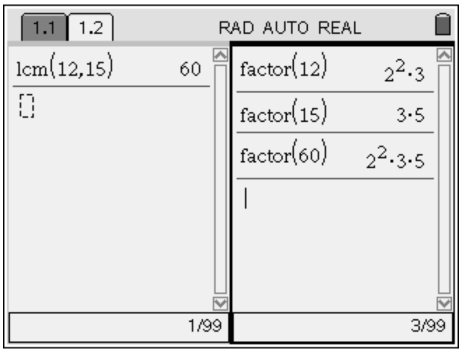
The first aim of this investigation is for students to establish an understanding of ‘lowest common multiple’. In this case, students are provided with a little less scaffolding. Teachers should invite students to volunteer pairs of numbers.

It is evident that students are willing to start ‘simple’. The lcm (2, 4) provided students with the first clue. The student that recommended lcm (6, 8) was already beginning to test their own conjectures. Such insightful suggestions from students should be noted.



A conclusion drawn very quickly by students after a brief investigation of the lcm function is that ‘it has something to do with the factors of the numbers’.

At this point it is useful to split the screen and use this environment to draw a number of comparisons between the result of lcm and the factor command.



Dividing students into groups and getting each group to investigate a range of numbers provides an opportunity to cater for the range of abilities in the class. Less able students may be supplied with significant scaffolding, while students who are more able, may be left entirely to their own resources.

This investigation also provides students with an opportunity to develop greater independence in future investigations.

Students need to see how the lowest common multiple is used in common calculations. The fraction capabilities of the calculator can be used to highlight the application of the lowest common multiple.

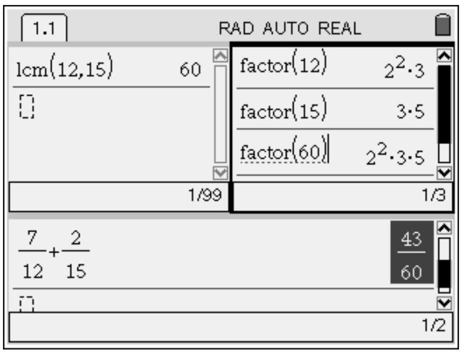
In the screen shot opposite, the three levels of the concept are displayed on different screens. This allows students to navigate between screens (and concepts) so that the various calculations and reputations are displayed around the screen.

Key questions here include:

How is the lowest common multiple determined?

How can the ‘factors’ help determine the lowest common multiple?

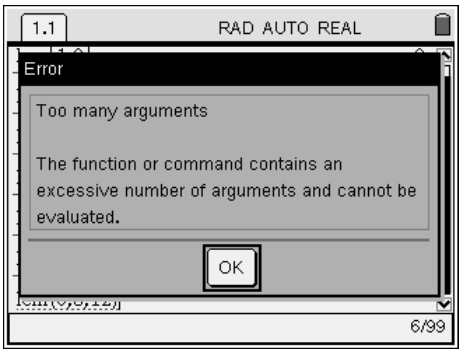
How is the lowest common multiple helpful for adding fractions?



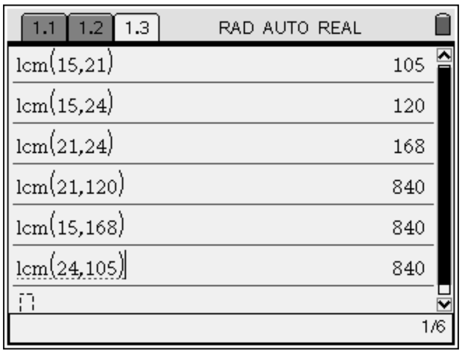
A useful limitation of the lcm command is the ability to accept only two numbers.

This leads to *simple* problems such as:

> Find the lowest common multiple of: 15, 21 and 24



Once students have *discovered* the limitations of the lcm command, they may choose *by hand* calculation as the most efficient approach. Alternatively, they may explore ways to use the calculator for such problems (shown opposite).



Questions of a much higher cognitive demand can be asked:

> Two numbers, *n* and *m* have only the factors 2 and 5 in common. Determine lowest common multiple of *n* and *m*.

Example 2 – index laws

The investigation into index laws included here is similar in concept to the surds investigation, in that it encourages students to undertake mathematical investigations. Several aspects of this investigation highlight common misconceptions that often lead to more significant errors later in a student's mathematical development. Consider the situation below as applicable to algebraic manipulation:

$$\begin{aligned} b\cdot(b-4) &= b \\ \frac{b\cdot(b-4)}{b} &= \frac{b}{b} \\ b-4 &= 1 \\ b &= 5 \end{aligned}$$

In the example illustrated here the solution $b = 0$ is lost. Of more significance, contradictions can occur where a division by zero is included in a *simplification* process:

$$\begin{aligned} 3x &= 2x \\ \frac{3x}{x} &= \frac{2x}{x} \\ 3 &= 2 \end{aligned}$$

In the example provided here, $x = 0$ is the only solution to $3x = 2x$, subsequently, the division by x leads to a contradiction. The index law investigation includes consideration of the law:

$$\frac{a^m}{a^n} = a^{m-n}$$

How does the CAS calculator handle this?

At a basic level, students can work with the simplifications of expressions such as:

$a \times a$ and $a \times a \times a$ as a^2 and a^3 respectively.

Students should be asked to write down results for:

$a \times a \times a \times a \times a \times a \times a$

or similar. Ask students to write their own rule that summarises how the calculator handles the above expressions. This is essentially a process of recognition of the relevant conventions and notation, and counting of the number of terms required for the index. It is important that students use their own words and understanding to identify the corresponding forms for expressions such as:

$a \times a \times a \times b \times b$

Students can then try other expressions such as:

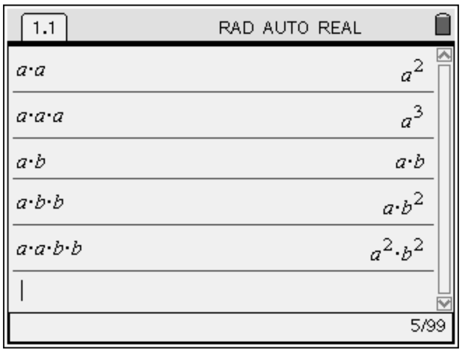
$a \times b \times b$ and also $a \times a \times b \times b$

and other combinations where the order of a and b terms in the product is mixed, for example:

$b \times a \times a \times b \times b \times a \times b \times b$

Once students have identified an appropriate set of *rules* and *conventions,* they should practise a range of examples so that they develop the mental facility for recognition of the correct result.

Given how rapidly students are able to simplify expressions such as: $a \times a \times a \times a \times a \times a$, they rarely use the calculator to perform such calculations or simplifications, as it simply takes too long to type the expression into the calculator.



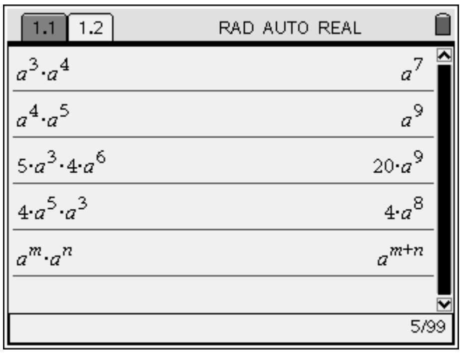
These concepts can be further explored through a range of specific and more general examples such as:

$a^3 \times a^4$ and $a^4 \times a^5$

Once students develop a ‘general’ rule, get them to see how the calculator treats the general situations:

$a^m \times a^n$ and $(a \times b)^n$

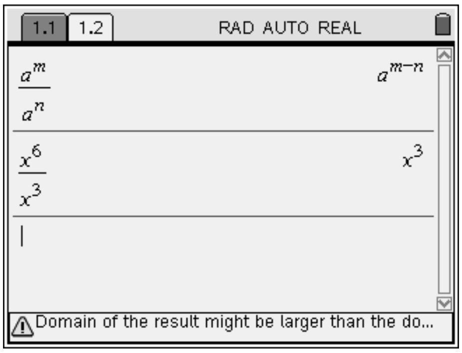
The two-dimensional layout editor enables a standard mathematical format to be used in entry, that is $a^3 \times a^4$ *rather than* a^3 × a^4. Navigation through the two-dimensional editor and templates is similar to that used in Equation Editor in Windows.



Another index rule to be introduced (often without exception) is:

$a^m \div a^n$

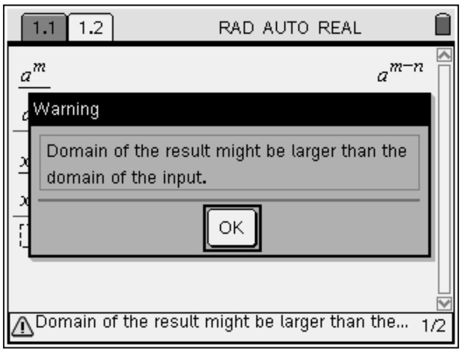
Teachers should highlight the warning at the bottom of the screen.



To view the warning message in its entirety, highlight the answer and press CTRL – Menu, which is equivalent to a *right mouse click* on a computer.

$$\frac{x^6}{x^3} = x^3 \ \ only\ if\ x \neq 0$$

Concepts often left to senior levels may need to be addressed earlier in the curriculum.



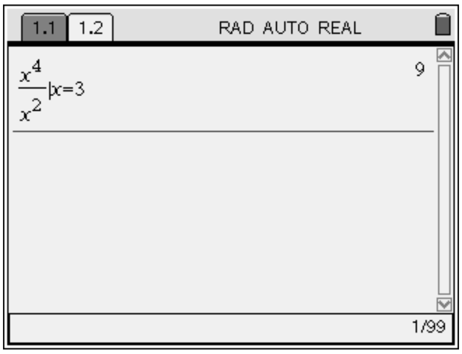
To help students understand this special case and further enhance their understanding of algebra, they should be encouraged to test rules with a range of numbers. This approach helps reinforce the concept that algebra is particularly useful in the expression of generality.

To substitute numbers into an algebraic expression use '|'. This may be *read* as 'given that'.

The expression shown can be *read* as:

'*x* to the power of 4 divided by *x* to the power of 2 given that *x* equals 3'.

This substitution into the simple expression provides a numerical model for students. Students should substitute a range of values for *x* to 'confirm' that their rule works.

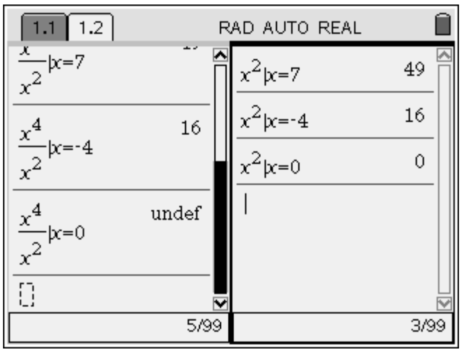


After trying a series of values, including negative numbers, fractions, decimals and even surds, students should try $x = 0$.

This substitution helps students recognise the limitation of the rule:

$$\frac{a^m}{a^n} = a^{m-n}$$

The split screen allows students to see the result in the original form and also in the simplified form.



It is left to the teacher to create challenging ways to investigate when, if ever, a rule breaks down. It reinforces generality but at the same time develops an insight for students to identify common causes such as:

- division by zero
- square root of negative numbers.

Example 3 – index laws: $12 \times 10^6 \div 6 \times 10^3$

Another source of error often encountered is connected to the *order of operations* as applicable to index laws and simplification. Consider the two situations below:

$$\frac{12\times10^6}{6\times10^3}$$

Compare this with:

$$12x^6 \div 6x^3$$

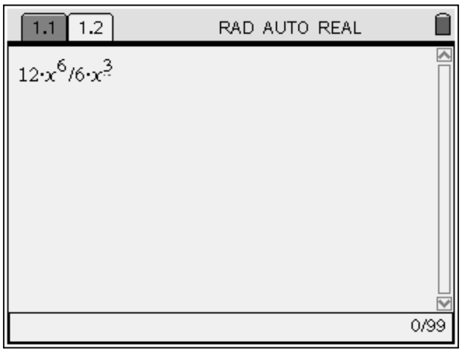
$$\frac{12x^6}{6x^3}$$

The numerical case: $12 \times 10^6 \div 6 \times 10^3$ is most evident when students in senior science classes continually produce erroneous results with respect to order of magnitude. Students need to be encouraged to develop expectations of magnitude when carrying out such computations. This example is a natural extension of the expression: $12x^6 \div 6x^3$, which is often erroneously simplified. While these two results can be compared *numerically* using a graphics calculator, specifically by substituting (storing) the value of 10 for *x*, the CAS makes the result obvious.

Encouraging students to use the two-dimensional editor provides an opportunistic return to *order of operations.*

Consider: $12x^6 \div 6x^3$

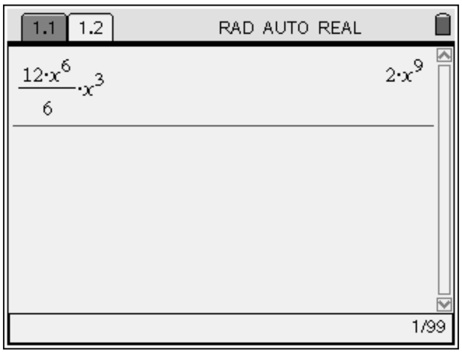
The calculator screen displayed opposite illustrates the *linear* method of entering the expression. This method has led to many errors, partially through student lack of clarity and understanding relating to the order of operations.



The calculator's interpretation of this entry is clearly displayed when it is executed (shown opposite).

The interpretation of the student-entered expression is clear and serves as a timely reminder. The output shown on the calculator indicates to students that they need to think carefully about order of operations, intended sequences of calculations, and the 'expected' output of a calculator.

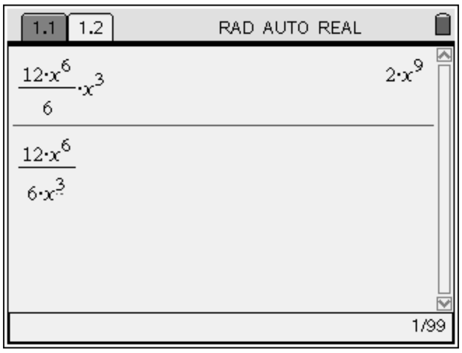
Students should be encouraged to discuss what they expected. This result highlights the importance of understanding the order of operations as well as acting as a trigger for discussing the importance of an expectation of an appropriate answer.



Contrast the previous results with the two-dimensional editing of the same expression.

As mentioned previously, the significance of this result is non-trivial.

In this context, students are developing good habits reinforced by the CAS calculator.



Year 9 program

Pythagoras' theorem

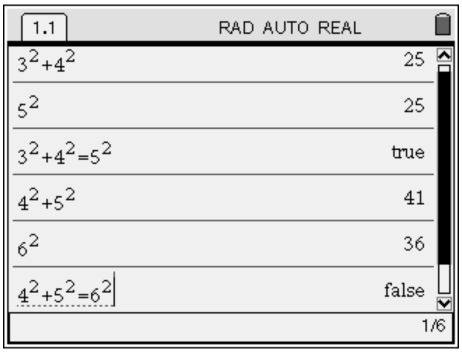
Timeline Semester 1	Level 6 Year 9 Topic	VELS Learning focus	CAS implementation
Weeks 7–9	**Pythagoras theorem** $a^2 + b^2 = c^2$ Calculate the length of the hypotenuse Pythagorean triples, including recognition of similarity	*Measurement, chance and data* • use Pythagoras' theorem to calculate the length of the hypotenuse • calculation and application of ratio • understanding the difference between exact and approximate *Working mathematically* • presentation of algebraic arguments using appropriate mathematical symbols and conventions • evaluation of the appropriateness of the results of students' own calculations • justification or proof of generalisations from specific cases • use technology to assist mathematical inquiry	Finding Pythagorean triples Example 1 More Pythagorean triples (extension) Example 2

Example 1 – finding Pythagorean triples

The first time students encounter the Pythagorean triple 3, 4, 5 of three consecutive numbers, they might try to extend this to other examples such as 4, 5, 6. It is important for teachers to ask students to test the validity of this proposal: does $4^2 + 5^2 = 6^2$. In this activity, students are encouraged to find another Pythagorean triple that consists of consecutive numbers.

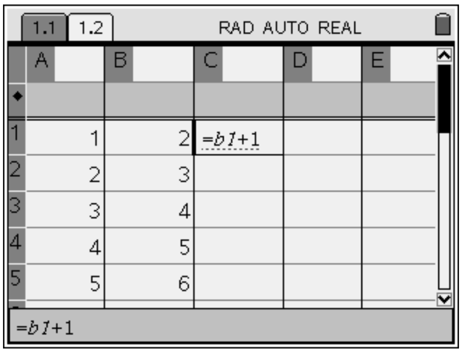
Students attempt to find other Pythagorean triples that consist of consecutive numbers by numerical guess and check techniques on the calculator page.

The Boolean true/false test capabilities of the calculator can also be used to check the validity of student guesses.



Spreadsheets are also used to progress towards an algebraic solution to this problem.

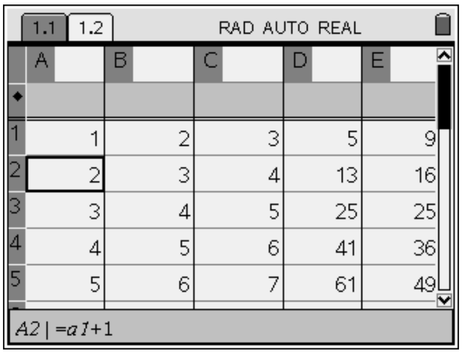
Spreadsheets work in a similar way to Excel™. Recursive formulas can be used to 'fill down'. Similarly, other sides on the triangle can be generated using formulas.



All the cells in the spreadsheet are based on the quantity placed in cell A1. Columns D and E are equal to $a^2 + b^2$ and c^2 respectively.

In the example shown here, row 3 shows that:

$3^2 + 4^2 = 25$ and $5^2 = 25$ therefore $3^2 + 4^2 = 5^2$



Changing the value in cell A1 means that the students can explore a greater range of values more quickly.

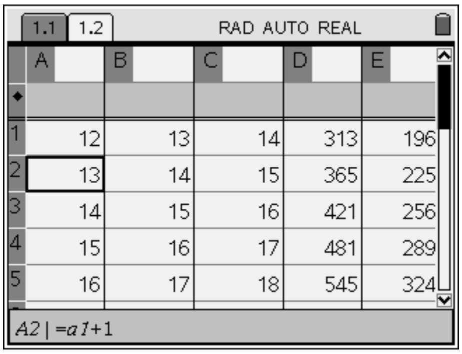
In this case:

$12^2 + 13^2$ and 14^2

$13^2 + 14^2$ and 15^2

$14^2 + 15^2$ and 16^2 …

have all been tested. After more and more testing, students begin to believe that their search may be futile.

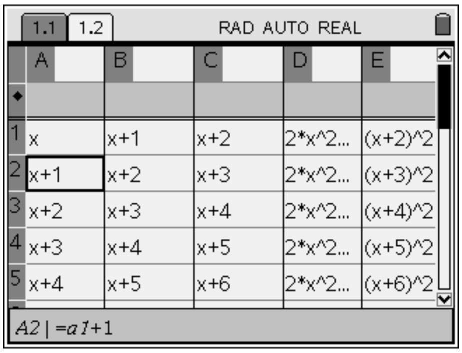


The spreadsheet can also deal with algebraic variables. Entering an x in cell A1 populates the spreadsheet with algebraic expressions. Of particular interest is the first row:

A1 = x

B1 = $x + 1$

C1 = $x + 2$



Returning to the home screen, students can use the solve command:

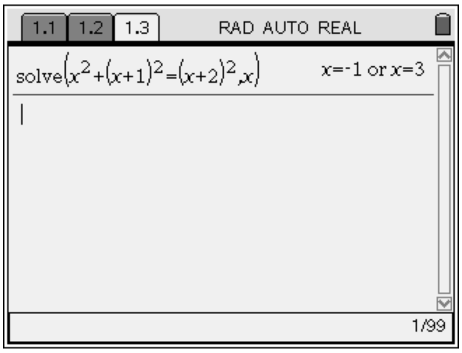
$$solve(x^2 + (x+1)^2 = (x+2)^2, x)$$

Students are required to interpret the solution to this problem.

$x = 3$ equates to the 3, 4, 5 Pythagorean triple.

$x = -1$ equates to -1, 0, 1.

While this second result is numerically accurate, it does not equate to a physical solution to the problem.



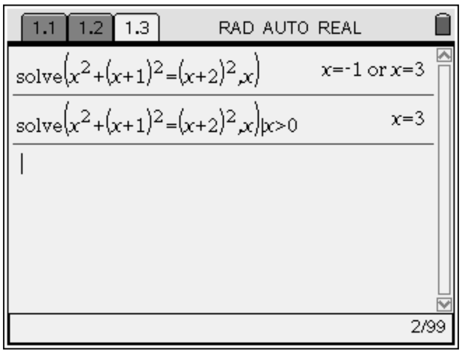
Students can follow this case with:

$$solve(x^2 + (x+1)^2 = (x+2)^2, x) \mid x > 0$$

The disadvantage of applying the restriction is that it narrows the student's necessity to interpret solutions from the original solution set.

The advantage of applying the restriction is that it introduces concepts related to trigonometric problems encountered further into the unit, such as:

$$solve(\sin(x) = \tfrac{1}{2}, x)$$



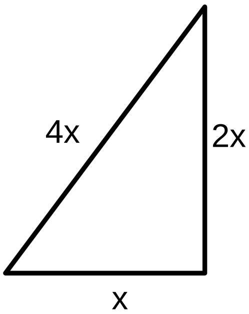
One of the most significant outcomes of this example is the use of algebra to solve a problem, and even more significantly, as a tool for proof. Furthermore, the problem in this example would not be accessible to most students early in Year 9. They are more likely to be working through expanding each of the quadratic expressions, manipulating the expanded expression and finally factorising the final expression. During these *processes* students often lose sight of what they are trying to prove.

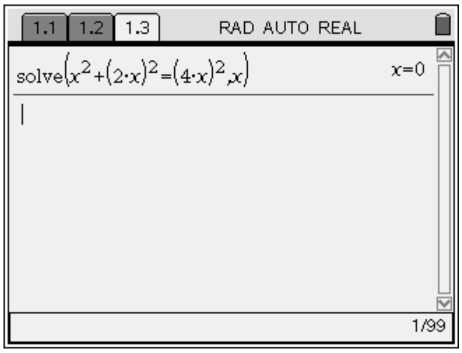
When students obtain the result $x = 3$, it needs to be stressed that this proves their conjecture … there are no other solutions to the problem: 'Are there any other Pythagorean triples consisting of three consecutive numbers?'

In this context, the CAS is being used to encourage students to *use* algebra as a problem-solving tool.

Following the search for other Pythagorean triples consisting of consecutive numbers, another problem is posed to students:

'Are there any Pythagorean triples where consecutive side lengths are doubled?'





Students generally proceed to formulate the following expression for evaluation:

$$solve(x^2 + (2x)^2 = (4x)^2, x)$$

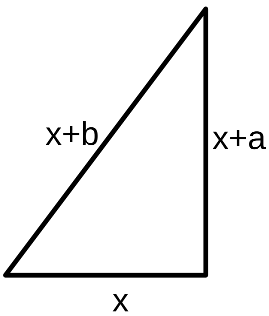
However, the corresponding empty set result should be used to reflect upon the original question. The most significant question that *must* be posed here is: 'Why are there no practical solutions?'

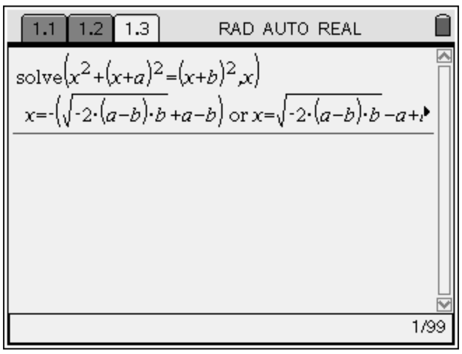
Looking at the diagram (above) and considering an inequality relationship between the three side lengths of any triangle, the practical reason for no solutions should soon become evident.

Example 2 – finding Pythagorean triples: an extension

Continuing the search for Pythagorean triples, the following provides students with a further challenge, using an algebraic approach, scaffolded by CAS. This approach focuses on the interpretation of the result.

Consider a the triangle below:





If this is a right angled triangle then:

$$x^2 + (x+a)^2 = (x+b)^2$$

If x, a and b are positive whole numbers and $a > b$ then a Pythagorean triple will be produced.

Student attention should be drawn to a section of the *answer*:

$$\sqrt{^{-}2 \cdot (a-b) \cdot b}$$

Which is equivalent to:

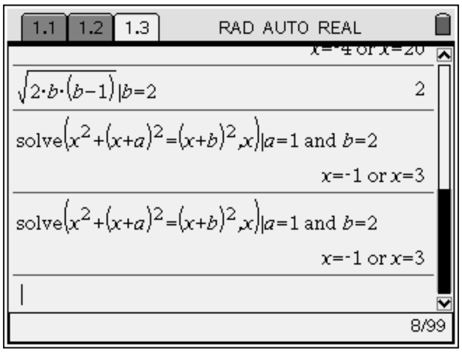
$$\sqrt{2 \cdot (b-a) \cdot b}$$

This portion of the answer produces an irrational solution (investigated in the unit on surds).

From this, students can be encouraged to explore situations where $a = 1$ and relate this to the original triangle '… the two shorter sides differ by 1'.

Systematic exploration quickly reveals a solution to this problem when $b = 2$; however, this amounts to the Pythagorean triple:

3, 4, 5.

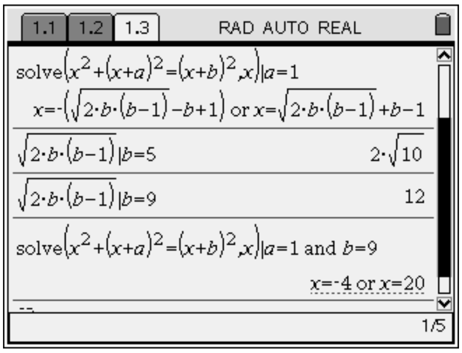


Further systematic exploration of b quickly results in a *new* Pythagorean triple when $b = 9$

Returning to the original expression, this provides the solution:

20, 21, 29

The two shorter sides differ by 1. The hypotenuse is 9 units longer than the shortest side.



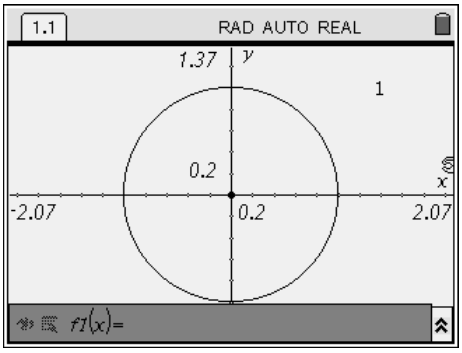
Year 9 program – measurement

Timeline Semester 1	Level 6 Year 9 Topic	VELS Learning focus	CAS implementation
Weeks 10–14	**Measurement** • The real number system • Surds operations +, - , ×, rationalise simple denominators	*Measurement* • Estimate and measure area • Estimate probabilities based on data • Generate data using experiments *Working mathematically* • Develop mathematical models to investigate and solve problems • Judge the reasonableness of results	Area of a circle Example 1 Area of a triangle Example 2

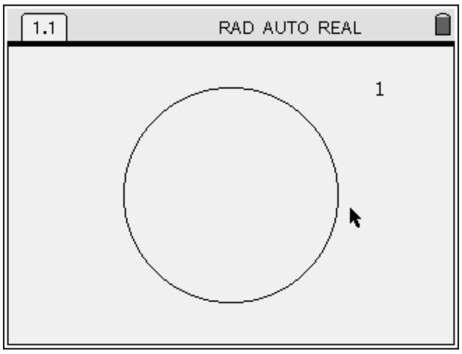
Example 1 – area of a circle (another piece of pi Monte?)

This is another example of *how* students learn being as important as *what* they learn. Students may be given the formula for finding the area of a circle and then asked to calculate the area of hundreds of circles. This activity focuses on finding a formula and using estimation skills to approximate the area of a circle.

The diagram that students use is constructed on the cartesian plane. Students begin by constructing a circle of radius 1 unit, centred at the origin.



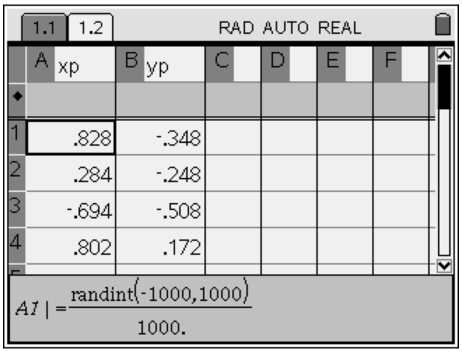
Hiding the axis and equation entry line produces the basis for the diagram to be used.



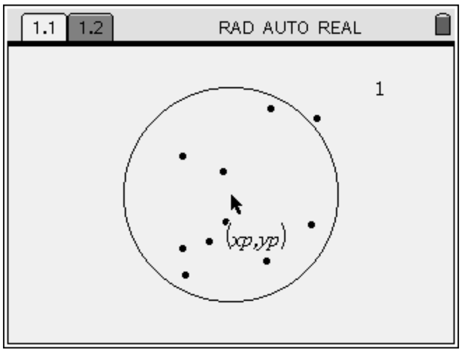
The spreadsheet is used to compute 10 pairs of random numbers between -1 and 1.

For simplicity, the RandInt command is used (as shown opposite). The columns are titled: Xp for the X point and Yp for the Y point to be plotted.

Just like a spreadsheet, the random numbers can be regenerated by pressing CTRL – R.



Plotting these points on the graphs and geometry page as a scatter plot provides a simulated environment.

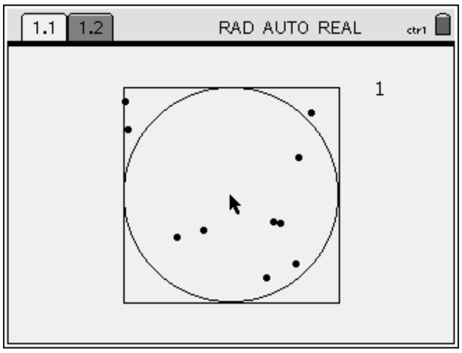


A square is drawn around the outside of the circle.

'What percentage of points do you think will fall inside the circle?'

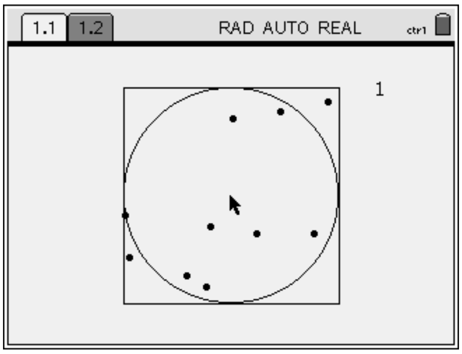
Students also need to be asked 'Why?' when responding to their estimate.

Typical student responses include: 'That's the amount of space the circle takes up in the square'.



Students repeatedly simulate the points and tabulate the results. Typical percentages for the proportion of points falling inside the circle after a reasonable number of simulations range between 75% and 85%.

Note: Students should be encouraged to practise a number of times before collecting data. Just like any other electronic device, the *random* numbers are only pseudo random numbers. The alternative is for students to see the random variable.



On the diagram shown opposite, the radius of the circle has been drawn in.

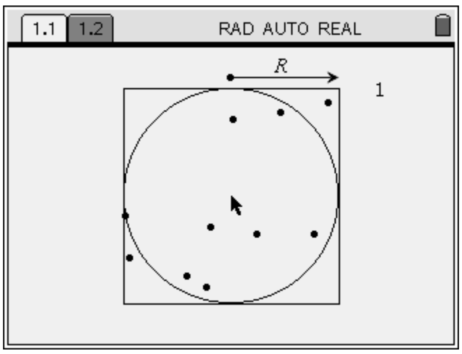
The square has side length 2R.

The area of the square is therefore: $4R^2$

Based on the simulation, the circle area is approximately 80% of the area of the square.

The area of the circle is therefore: $0.8 \times 4R^2$

Simplifying this expression: Circle Area $\approx 3.2R^2$



Typical introductory questions for students after the completion of this activity include:

> A square has an area of 120 cm^2, what would be the area of a circle that just fits inside the square?
>
> A square measures 20cm × 20 cm. Estimate the area of a circle that just fits inside the square.

The concept can be extended to three-dimensional shapes:

> A cylinder just fits inside a cube of 1000 cm^3. What would be the volume of a cylinder that just fits inside the cube?

The importance of *expectation* of a suitable answer is then extended to an unfamiliar situation:

> A sphere just fits inside a cube of 1000 cm^3. What would be a reasonable measurement for the volume of the sphere? (Justify your answer)

This last question illustrates the importance of engaging students in these types of activities as they provide students with an experience to draw upon.

Year 9 Program – structure: algebra

Timeline Semester 1	Level 6 Year 9 Topic	VELS Learning focus	CAS implementation
Weeks 15–18	**Creating and solving algebraic equations** • writing and solving algebraic equations • solving algebraic equations (Using the TI-92Plus Symbolic Maths Guide) • solving algebraic equations by hand	*Structure* • factorisation of an algebraic expression by extracting a common factor *Working mathematically* • evaluation of the appropriateness of the results of their own calculations • presentation of appropriate arguments using appropriate mathematical symbols and conventions	Paving problem Example 1 Vedic mathematics Example 2

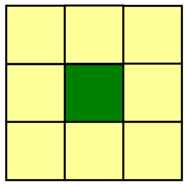
Example 1 – paving problem

This problem is often presented to students as the ‘paving problem’. In a written document, such as this, the content is not as powerful. In class, this activity is *always* accompanied with a Microsoft PowerPoint presentation that leads students into a particular way of thinking. The animations within the presentation provide a stimulus for the algebraic expressions formulated by students. A static version of the presentation provides some idea on how students are led into a particular chain of thought. Students’ responses (summarised here) are representative of students’ contributions as they explain the reasoning behind their mathematical expressions for the number of pavers.

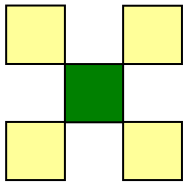
The problem is to provide a set of pavers around a series of square garden beds. The first in the series is a 1 m × 1 m (or 1 m^2) garden bed.

Stage 1

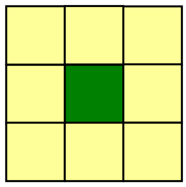
The first garden bed is illustrated here. There are 8 pavers around the garden bed.



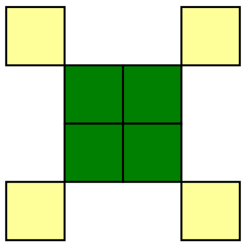
The animation on the Microsoft PowerPoint shows the corners being laid first (as shown). Note that this will be the same for all sized garden beds, four pavers in the corners.



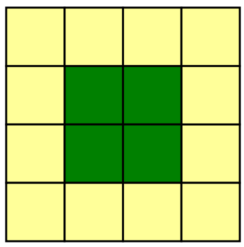
The animation then includes the next 4 pavers being laid, one on each side of the garden bed.



The second garden bed is a 2 m × 2 m garden bed, as an animation the four corners appear first, as before (as shown).



With the four corners appearing first, 8 more pavers are added, two on each of the four sides.



By the time the students have seen the third garden bed, a 3 m × 3 m garden bed, they have begun to formulate an algebraic expression for the number of pavers to be laid for an $n \times n$ garden bed:

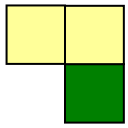
Number of pavers, P, for an $n \times n$ garden bed is $P = 4 + 4n$

Students are then required to relate their *rule* to how the pavers are being laid. Typical responses from students include: 'You lay the four pavers on the corners, then for the n units of length on each side of the garden you need to lay n pavers. There are four sides so that is $4n$ pavers more. So the rule is $4 + 4n$.'

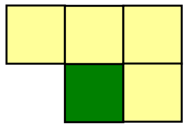
Stage 2

The paving pattern is exactly the same as the one provided in Stage 1. However, the pavers are presented in a different order. Students are required to describe the number of pavers based on the way they are being *laid*.

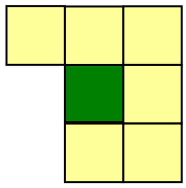
The garden bed is displayed, followed by two pavers (as shown).



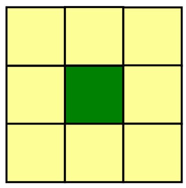
The part of the animation includes two more pavers.



The third stage of the animation shows another two pavers being laid.



The final stage of the animation shows the completed pattern.



Students generally require the first three animated diagrams before they can establish a rule. In this case the rule they typically produce is:

$P = 4(n + 1)$

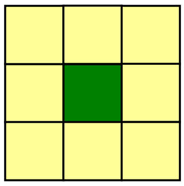
Once again students are asked to explain how their rule was determined. Typical responses include: ‘Four **sets** of pavers are laid down each time. The number of pavers laid on each side is one more than the garden bed. So the rule is 4 lots of $n + 1$ pavers; or $P = 4(n + 1)$’.

Once again the animation has provided a way of thinking about the formulation of the algebraic expression.

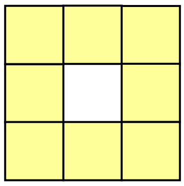
Stage 3

Although this animation is the simplest, it usually poses the greatest challenge for students. Again, usually by the fourth animation most students generate a formula for the amount of pavers.

In this case, the animation begins with all the pavers present.



In the animation, the centre square, the garden itself, disappears.



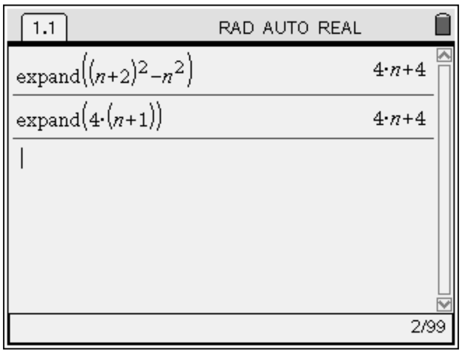
In this case an appropriate formula is:

$$P = (n + 2)^2 - n^2$$

Again, students are asked to explain how their rule was determined. Typical responses include: ‘You start with side lengths that are 2 pavers longer than the garden bed. This forms a square, so there are $(n + 2)^2$ pavers, but then you take away the garden bed which is n^2; so the rule is: $(n + 2)^2 - n^2$ for the number of pavers’.

Due to the physical nature of this problem, it is easy to convince students that all three rules are simply different representations of the same quantity (the total number of pavers). However, each rule represents a different means of expressing the number of pavers.

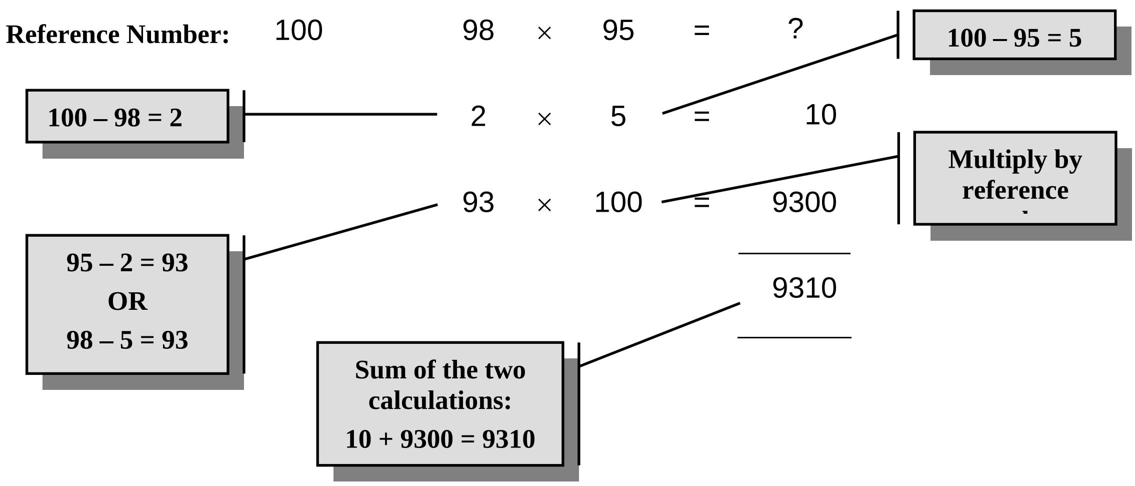
Students use the ‘expand’ command to show that all three representations are the same.



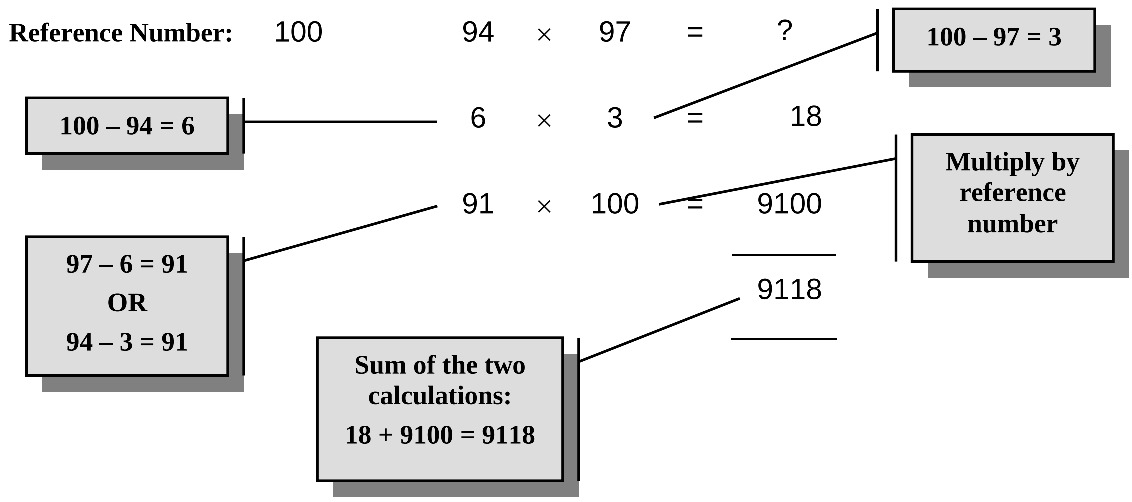
This activity represents one of many where students use can use ‘expand’ or ‘factorise’ to show the equivalence of various algebraic expressions.

Example 2 – Vedic mathematics

One of the interesting uses of Vedic mathematics is a technique used to multiply numbers. Rather than using *traditional* multiplication techniques, some time is spent showing students how to multiply two numbers, specifically two numbers close to 100, for example 98 × 95 as illustrated below. Students can generally master the technique for a small range of numbers within 10 to 15 minutes. At this point, some students generally ask: 'Does this always work?', and 'Why haven't we been taught this before, its much quicker and easier?'



Another example:



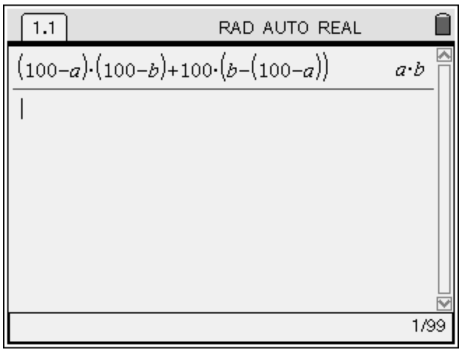
An algebraic approach, an answer to: 'Does it always work …?'

Reference Number:	100	a	×	b	=	
		$100 - a$	×	$100 - b$	=	$(100 - a)(100 - b)$
		$b - (100 - a)$	×	100	=	$100.(b - (100 - a)$

Sum of the two calculations is completed on the calculator. Students are required to predict the outcome.

The two expressions are added together on the calculator. Students are required to interpret the answer.

Students are generally *surprised* by the simplicity of the result. Students that are more able like to know, ‘What happened to all the other bits?’ To answer this question, the students need to be able to expand *by hand* to see how particular components cancel each other out.



Appropriate questions to ask students are:

Does it work if the numbers are bigger than 100?

What values would the variables *a* and *b* take on for 103 × 105?

By studying the algebraic expressions, suggest a technique for multiplying numbers bigger than 100.

If the reference number was changed to 10, and numbers close to 10 were chosen, would this technique still work? If so, prove it!

Year 9 program – linear graphs

Timeline Semester 2	Level 6 Year 9 Topic	VELS Learning focus	CAS implementation
Weeks 4–7	**Linear graphs** • coordinates of points • gradient of straight lines • *y* = *mx* + *c*	*Structure* knowledge of the quantities represented by the constants *m* and *c* in the equation: y = *mx*+*c* *Working mathematically* use technology to assist mathematical inquiry	Coordinates of points Example 1 General equation to a straight line Example 2

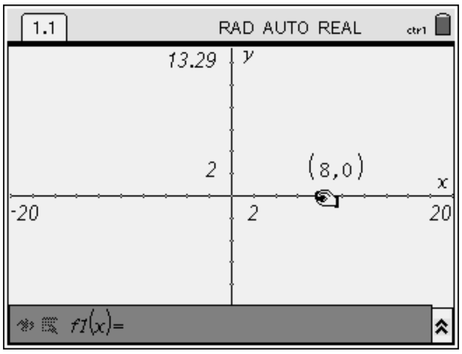
Example 1 – coordinates of points

The inclusion of this activity is to review how points are represented on the cartesian plane and highlight particular points such as points on the *x* or *y* axis.

Using a graphs and geometry page, students put a point on the *x* – axis and measure the coordinates of the point.

Students drag the point around (along the *x* – axis) and respond to two basic questions:

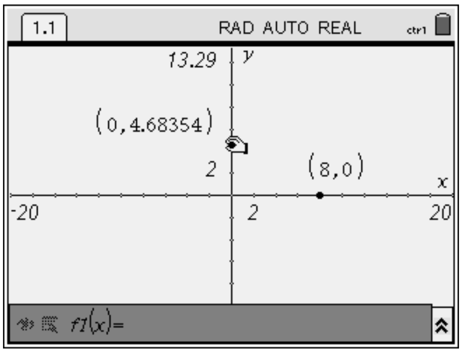
- Write down the coordinates of 5 points that are on the *x* – axis.
- What do all of these points have in common?



Students put a point on the *y* – axis and measure the coordinates of the point.

Students drag the point around (along the *y* – axis) and respond to two basic questions:

- Write down the coordinates of 5 points that are on the *y* – axis.
- What do all of these points have in common?

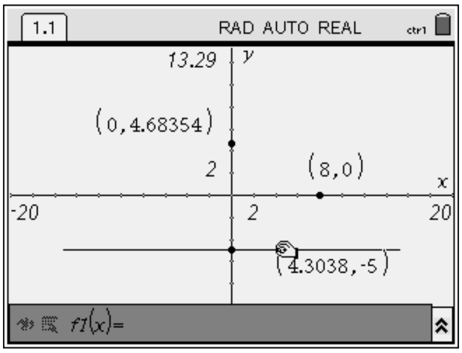


The task is for students to construct a line parallel to the *x* – axis and plot a point on this line.

Students drag the point around (along the line) and respond to two basic questions:

- Write down the coordinates of 5 points that are on the line.
- What do all of these points have in common?

While these activities may seem primitive, they form an important part of the subsequent activity that asks much deeper conceptual questions. It is important that students have a clear understanding of these basic principles.



The task is for students to construct a line parallel to the y – axis and plot a point on this line.

Students drag the point around (along the line) and respond to two basic questions:

- Write down the coordinates of 5 points that are on the line.
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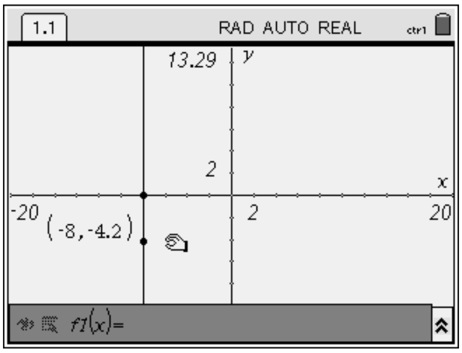
While these activities may seem primitive, they form an important part of the subsequent activity that asks much deeper conceptual questions. It is important that students have a clear understanding of these basic principles.



At the conclusion of this activity, students are required to make their own notes, summarising what they have learned. Some students do not write anything as they are familiar with this information; other students busily write down information relating to all four scenarios.

This activity encourages students to take responsibility for their own learning, a skill further utilised in the next activity.

Example 2 – general equation of a straight line

Students begin with the Graphs and Geometry page and plot a point on the y axis and then measure the coordinates of this point. Students move this point to a pre-defined location: (0, 5)

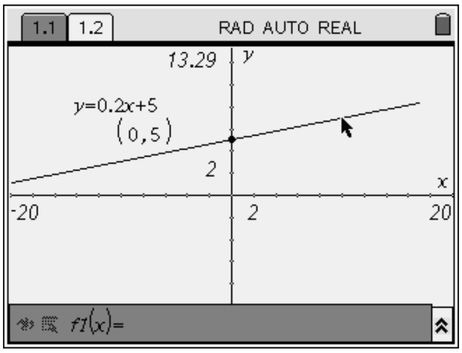
The next step is for students to draw a line through this point and measure the equation to this line.

Appropriate questions at this point:

- Write down the equation of three lines that pass through the point (0, 5)
- What does each of these equations have in common?

In response to student answers '… the number without the *x* tells you where it crosses the *y* – axis'.

- Why?

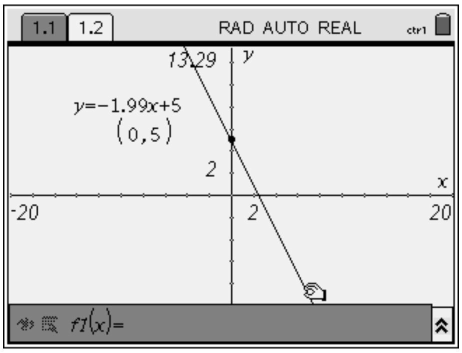


Following this activity, students are instructed to place the point on the *y* – axis at a new location and write down an equation that passes through this point. Students share their equation with the rest of the class. The class is then required to work out the coordinates of the point the line passes through.

Following this activity, students investigate further the slope of the line.

- Draw a graph of a line when the number in front of the *x* is negative.
- Draw a graph of a line when the number in front of the *x* is positive.
- What does the equation to the line look like when it is horizontal? Why?

Note: Conduct this activity after students have completed the distance time match using the Texas Instruments CBR (Calculator Based Ranger). This activity develops meaning for a positive slope, negative slope, zero and infinite slope.



Following this activity, students are engaged in an extension to the *distance – time match* activity. Using the CBR students are asked to *walk* particular graphs:

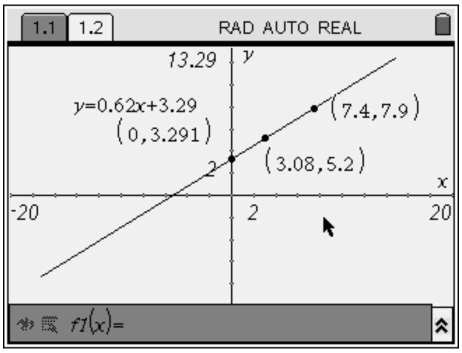
- Walk a graph of the equation: $y = x + 1$
- Walk a graph of the equation: $y = 0.5x + 2$
- Walk a graph of the equation: $y = -0.3x + 3$
- Walk a graph of the equation: $y = 2 – x$

During this time, other activities are conducted, including the using calculator robots, match stick patterns and a range of other data collection activities. The aim is to establish understanding of the two parameters *m* and *c*.

A further extension of the previous investigations develops a more formal treatment of gradient.

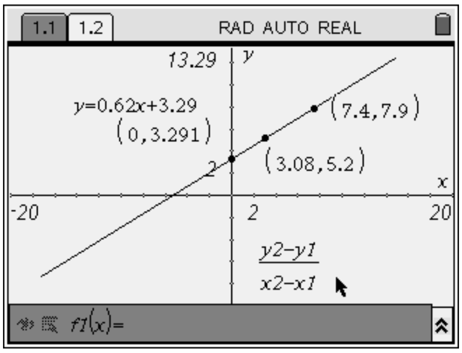
Two points are placed on a line drawn on the cartesian plane.

The coordinates of these points are measured.

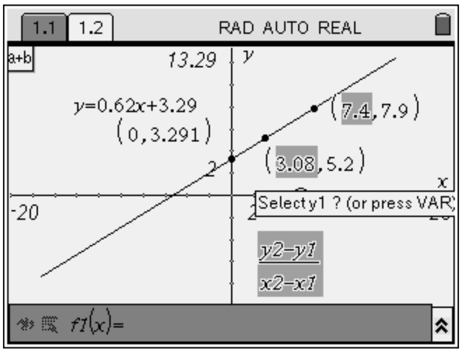


Students use the text tool to write the formula:

$$gradient = \frac{y_2 - y_1}{x_2 - x_1}$$



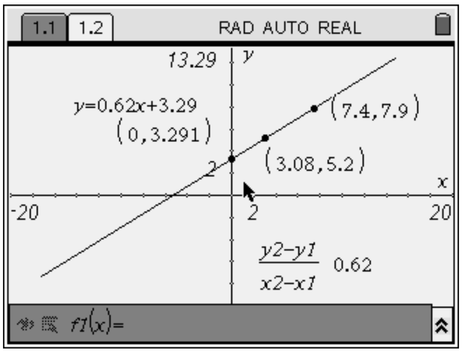
The calculate tool allows text expressions to be computed. This is a very powerful tool conceptually. The selection of the expression leads to a *request* via prompts of each variable in the expression (as shown).



The expression is evaluated and compared with the equation to the line.

Students drag the two points on the line backwards and forwards along the line and note that no change occurs to the computed gradient. Students understand from this activity that the gradient of a straight line is constant.

This concept is re-visited, later, using difference tables and compared with the matchstick activities, where the focus is on adding *m* matches each time the pattern is incremented.



Year 9 program – trigonometry

Timeline Semester 2	Level 6 Year 9 Topic	VELS Learning focus	CAS implementation
Weeks 12–14	**Trigonometry** • construct a table of values for a range of ratios in a right angled triangle • use trigonometric tables to estimate the size of unknown lengths and angles in a right triangle • use technology to determine the size of unknown sides and angles in a right angled triangle	*Measurement* • use trigonometric ratios to calculate the length of unknown sides in a right angled triangle *Working mathematically* • develop mathematical models to investigate and solve problems • judge the reasonableness of results	Constructing ratios Example 1

Example 1 – constructing ratios

This activity extends from very early days of using *Geometer's Sketchpad*. It arose out of the frustrations students were experiencing when trying to work out how to calculate the size of an angle using trigonometry. Students often find it difficult to determine when to 'press inverse sine' and when to 'just press sine'. This indicates that the students do not understand what they are doing – not just from a calculation point of view, but when their focus is purely on determining the order in which to *press buttons*. Other experiences, particularly relating to transposing equations incorrectly, highlighted a lack of student understanding. This is often seen when students are required to determine the length of the hypotenuse in a *sine* calculation.

$$\sin(40) = \frac{10}{hyp}$$

All too often the next line of student working out produces:

$$hyp = 10 \times \sin(40)$$

Students then happily write down the length of the hypotenuse as: 6.43

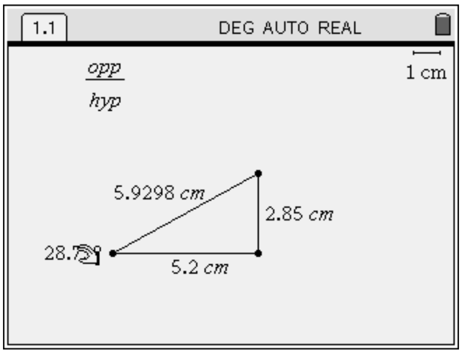
At this point, consider the statement: 'Students judge the reasonableness of their answer'.

Students begin using the Graphs and Geometry application and opting for a 'Plane Geometry View'.

Students learn how to construct a right angled triangle and measure the side lengths and one of the angles.

The text tool is used to write the expression for the ratio:

$$\frac{opp}{hyp}$$

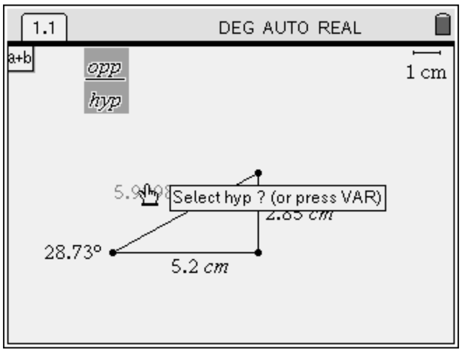


Using the calculate tool, students measure the size of the ratio, prompted by the calculator to identify the 'opp' side and the 'hyp'.

The next is instruction is for students to adjust the triangle until the angle measures 30°. Students then contribute values for the ratio. Ask students whether they *copied,* since all the ratios seem to be about the same size. This discussion leads to students drawing their triangles on the board with the respective side lengths filled in. Students tend to respond: 'It doesn't matter how big the triangle is, the ratio stays the same'. This observation is then investigated for a range of other angles; subsequent *clarifications* result in a concise and useful contribution from students about the *sine* ratio.

Armed with this information students are required to construct a table of values for a range of angles and ratios. Questions are posed requiring students to estimate the size of an angle or ratio given appropriate information.

Other more conceptual questions are asked such as: 'Why doesn't the ratio ever exceed '1'?'



During the course of this activity, students produce a table of values for the trigonometric ratios: sine, cosine and tangent. On occasions, students have also produced tables of values for secant, cosecant and cotangent. Recall the difficulties students face when computing the hypotenuse, knowing these other ratios eliminates this problem. Access to cosecant (csc), secant (sec) and cotangent (cot) on the calculator make teaching these ratios a reasonable thing to do, particularly since the calculator can call up all of these ratios.

During initial questioning involving the determination of side lengths and angles, students do not realise they have constructed ratios that the calculator can compute. Students are required to estimate angles and side lengths based on their table of values (increments of 10° are sufficient for the tables of values).

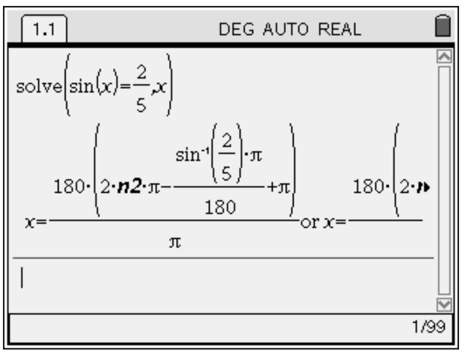
After some introductory use of trigonometric ratios and relatively informal algebra, students are told about trigonometric ratios: sine(x), cosine(x) and tangent(x). Students learn to estimate and look up tables. Furthermore, students are required to estimate the size of the angle or side length before looking up the tables, which becomes a *challenge* for students to improve their accuracy.

It is ironic that students are quite happy to use their constructed table of *ratios* when a *perfectly useful* calculator is available. The success of this activity is evident, exemplified by, one of the most insightful comments from a student: ‘Do we have to use the calculator in the test or can we use our tables?’ The first point here is that the student wants to use the table of values. The second point is made when the student is asked why they should reject such technology, and replies: ‘I don’t get it when I have to press the buttons on the calculator; I prefer to use my tables’.

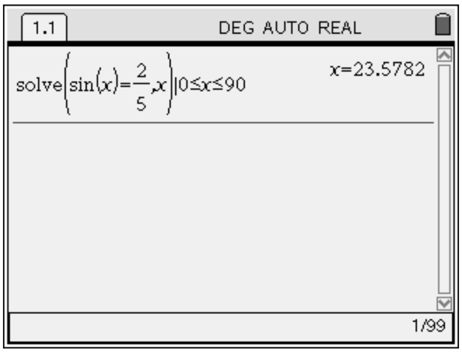
Highlight the difference between *sine(x)* and *sine*$^{-1}$(x) by focusing on whether students need to look up the table *forwards* or *backwards.* Finally, introduce those students that are only able to identify the correct ratio, to the solve command. The biggest problem with this method is when students are required to calculate the angle; they need to restrict the range of angles, making the problem almost as complicated.

If students attempt to determine the size of the angle using the solve command:

The calculator is a very sophisticated tool and generates the sequence of answers, that is, a form for potentially listing all solutions.



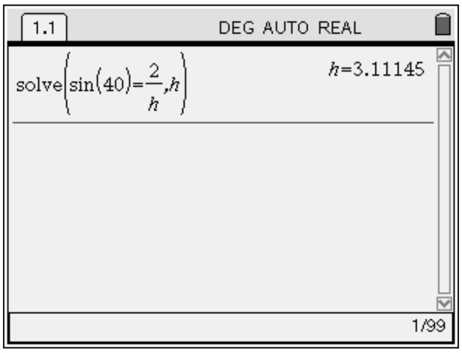
If the students restrict the angles to: $0 \le x \le 90°$ the answer is more usable:



However, the solve command is useful for students trying to determine the length of the hypotenuse in cases such as:

$$\sin(\theta) = \frac{opp}{hyp}$$

$$\cos(\theta) = \frac{adj}{hyp}$$

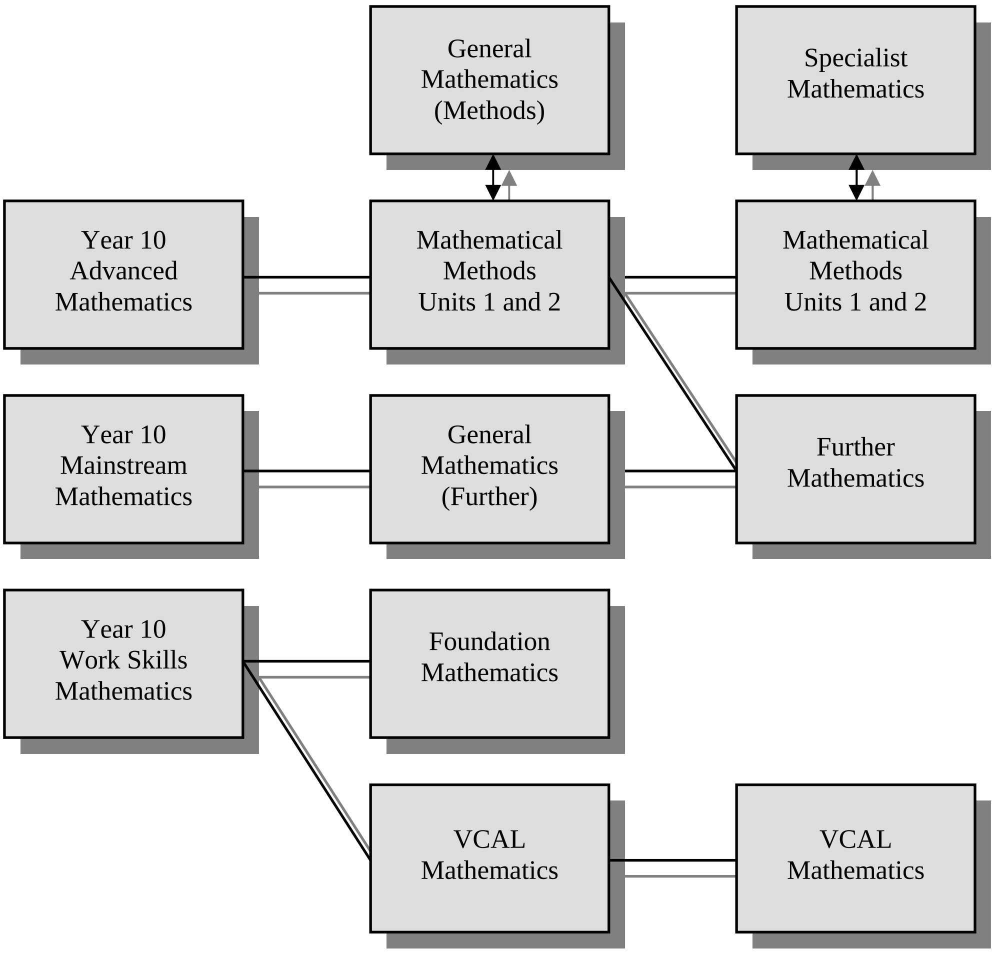


In such cases, reports to parents include comments: '… can determine unknown lengths and angles in a right angled triangle with the assistance of a CAS calculator'.

There is an acknowledgment that students are able to recognise the appropriate ratio from a diagram.

Year 10 program
Mathematics curriculum – Elisabeth Murdoch College

The Year 10 mathematics curriculum at Elisabeth Murdoch College has been structured to cater for the range of abilities, interests and aspirations of students. A significant number of students in Year 10 elect to study a Year 11 (Units 1 and 2) VCE subject. These Year 10 students come under the remit of the senior school. Subjects are blocked on the timetable to provide maximum opportunities for students to coordinate the VCE and Year 10 study combinations. Students entering Year 10 are provided with three main options, reflective of the mathematics options provided in VCE Units 1 and 2, as shown below. Students studying Advanced Mathematics (preparation for Mathematical Methods Units 1 and 2) in 2008 were required to purchase a CAS calculator (TI-*n*spire CAS). Students studying Mainstream mathematics were welcome to purchase a CAS calculator, but the entry level calculator for this subject continues to be a scientific calculator. A class kit of TI-*n*spire CAS calculators is available for teachers to use with these classes.



These pathways are the most common pathways selected by students. It is important to note that students in the Advanced Mathematics class in Year 10 may complete any of the following pathways:

- Mathematical Methods Units 1 and 2 (Year 11) then Mathematical Methods Units 3 and 4 (Year 12)
- Mathematical Methods Units 1 and 2 and General Methods Units 1 and 2 (Year 11) then Mathematical Methods Units 3 and 4 (Year 12)
- Mathematical Methods Units 1 and 2 and General Methods Units 1 and 2 (Year 11) then Mathematical Methods Units 3 and 4 (Year 12) and Specialist Mathematics Units 3 and 4 (Year 12)
- Mathematical Methods Units 1 and 2 and General Methods Units 1 and 2 (Year 11) then Mathematical Methods Units 3 and 4 (Year 12) and Further Mathematics Units 3 and 4 (Year 12).

The materials outlined here are excerpts from the Year 10 ‘Advanced Mathematics’ course. From the pathways provided above and associated VELS content, the course must provide for a range of topics; however, many of the topics are done at a more *advanced* level. This structure was implemented at the beginning of the 2007 school year.

Broad topic block planner

Year 10 (advanced mathematics)			
Semester 1		**Semester 2**	
Weeks	**Topic**	**Weeks**	**Topic**
1–6	Statistics and probability	1–5	Quadratic functions
7–10	Algebra and equations	6– 7	Exponential functions
11–13	Number systems	7–10	Systems of equations
14–16	Linear equations	11–13	Circular functions
17–18	Revision and examination	14–16	Geometry
		17–18	Revision and examination

Year 10 program – statistics and probability

Timeline Semester 2	Level 6 Year 10 Topic	VELS Learning focus	CAS Implementation
Weeks 1–6	**Statistics and probability** • construct	*Measurement, chance and data* • represent statistical data using technology • estimate probabilities using data • calculate summary statistics for data including mean / mode / median *Working mathematically* • create a simulations using technology • formulate and test conjectures • use technology to assist in mathematical inquiry	A simple dice game Example 1

Example 1 – a simple dice game

The object of the game is to move 20 squares, but you must ‘roll’ the exact number to finish the game. If you could choose a multi-sided dice, how many sides would you like? Players can choose their own dice. One player could use a 6-sided dice while their opponent uses a 20-sided dice.

Initially, students believe a 20-sided dice would be the best. The calculator’s random number generator allows students to determine exactly how many sides they would like on their dice and also provides a record of their rolls.

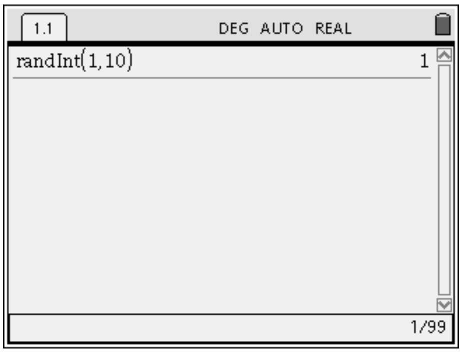
The calculator page is the place to start for this simulation.

Choose the number of sides for your dice.

The example shown here is for a 10-sided dice.

Type in: randInt(1,10)

Students play a number of games and record their findings. Over time they get a sense of how many sides are appropriate to reduce the number of turns.

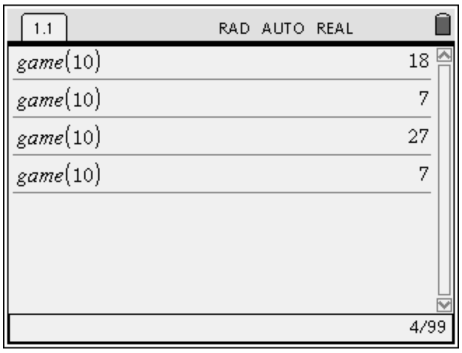


After students have developed an idea of how the game is played, they become focused on the problem:

What is the optimum number of sides for the dice in order to reduce the number of rolls required and hence win the game?

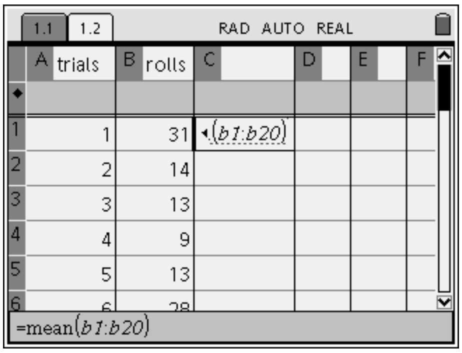
A simple program (shown here) allows students to simulate an entire game in a split second with a nominated number of sides on the dice.

The screen opposite displays the results of four games where a 10-sided dice has been used: 18 rolls for the first game, 7 for the second game, 27 for the third, and 7 for the last.



The next step is to combine the spreadsheet and the games program (function), and simulate multiple games at once.

The spreadsheet can also display the average of the 20 games being simulated here. This is plotted as a dynamic line to provide a sense of the *expected number* of rolls in order to win the game.

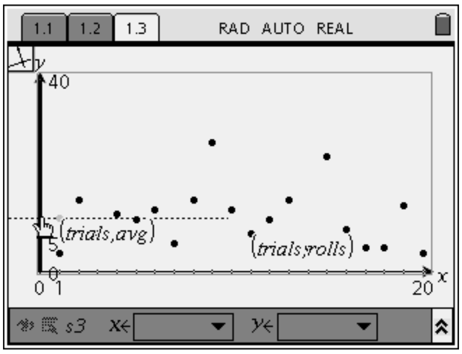


The screen shown here displays the results of 20 games using a 10-sided dice; the *average* number of rolls is being plotted (dotted) line.

Another 20 simulations can be generated by pressing CTRL-R (for recalculate). This gives a sense of how the number of sides on the dice affects the duration of the game.

Students can change the number of sides on the dice and instantly view the change in the results, and then average out these results.

The *variation* of results is one of the points clearly displayed. When students choose a 20-sided dice, some games are completed in one to two moves, while other games take fifty or more moves. In contrast, using a 6-sided dice, *most* games are completed in around ten moves.



There are a number of important concepts developed in this investigation:

- The theoretical probability is quite complicated in this activity. The simulations give students a very good understanding of the optimum solution.
- The graphical display introduces concepts such as *variation* leading to a discussion on *standard deviation* in an informal context (although this figure could be calculated using the spreadsheet).

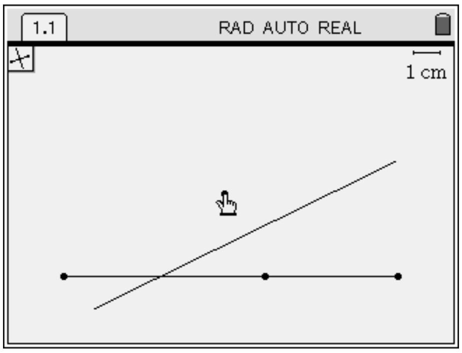
Year 10 program – quadratic functions

Timeline Semester 2	Level 6 Year 10 Topic	VELS Learning focus	CAS implementation
Weeks 1–6	**Quadratic functions**	*Structure*	
	• construct parabolas using paper folding techniques	• represent quadratic functions by rule, table and graph	What is a parabola? Example 1
	• develop an understanding of the properties of a parabola	• recognise and explain the roles of *a*, *b* and *c* in $f(x) = a(x + b)^2 + c$	The discriminant Example 1
	• understand the connections between the various algebraic representations of a quadratic equation	• distinguish between types of functions using difference tables	Transformations Example 3
	• understand the connections between the algebraic representation of a quadratic equation and the corresponding parabola	• use and interpret functions in a range of modelling situations	
		Working mathematically	
	• factorise quadratic equations using technology and by hand	• use technology in various combinations to assist in mathematical inquiry	
	• model the motion of a bouncing ball or the path of a projectile (using data logging equipment)	• analyse functions and carry out symbolic manipulation	
		• use geometry software or graphics calculators to create geometric objects and transform them	

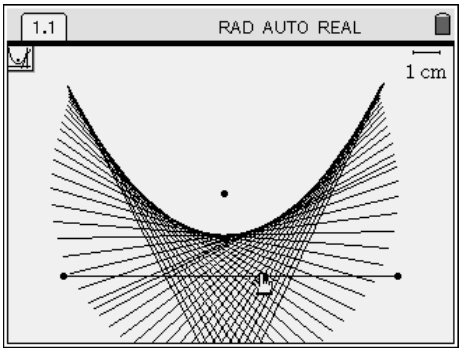
Example 1 – what is a parabola?

This activity is simple, quick and brings about an understanding of a parabolic curve. Students begin by folding paper in such a way as to create an ‘envelope’ in the shape of a parabola. In most cases, prior experience results in one or more students identifying the envelope as a parabola. This paper folding is then reproduced on the calculator using the dynamic geometry environment. Connections are drawn between the paper folding experience and the calculator dynamic geometry environment.

The segment at the base of the screen represents the base of the page from the student’s paper folding experience. The perpendicular bisector represents the crease in the paper folding experience.

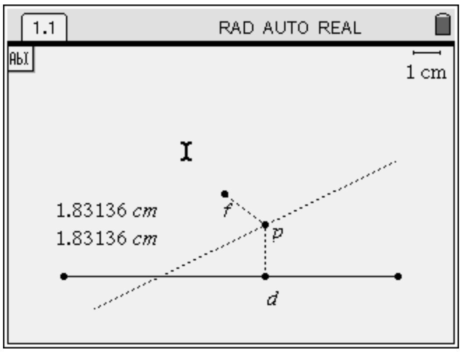


A locus of the perpendicular bisector generates the envelope students have created through the paper folding exercise. The advantage of this environment is that students can move the 'point' (focus) and line segment (directrix) to see the effect on the envelope in a dynamic way.



Students use some of the construction tools to identify the point P; a point that is equidistant from points f and d. This measurement is displayed opposite.

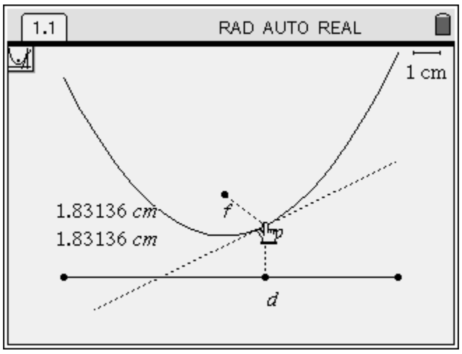
Able students will show why these distances must be the same (congruent triangles), thereby fulfilling other components of the mathematics dimensions.



A locus of point P creates a *special curve*. This curve is described to students as a *parabola*. Students drag point f around the screen and observe the effects it has on the parabola (geometric transformations including dilations, translations and reflections).

Other activities are included to reinforce the definition of a parabola. These activities include instructing students to go and stand at a point that is equidistant from the back wall of the classroom and the teacher.

The students form a curve, a parabola.



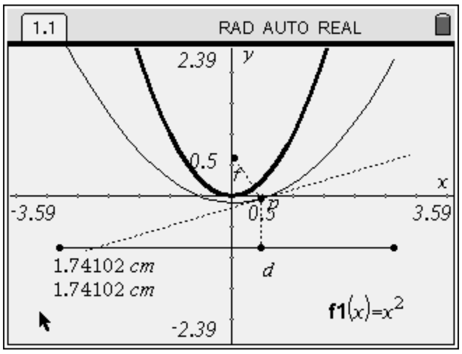
The geometry environment in TI-*n*spire CAS interacts with the cartesian plane.

Students are asked to graph the equation: $y = x^2$

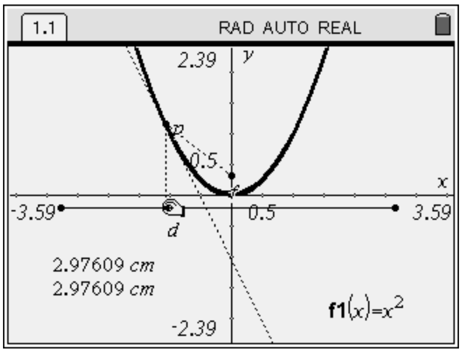
Students are required to drag the line segment (directrix) and point f (focus).

Can you fit this curve to the parabola?

Note: It is possible to drag the parabola $y = x^2$ around the screen; however, this changes the equation. The object here is for students to see that the basic form of a quadratic equation: $y = x^2$ fits the geometric description.



After some manipulation the locus can fit very neatly over the parabolic curve y = x2. Dragging point d along the original line segment reviews the concept that a parabola is a set of points equidistant from a fixed point and a line.



One of the driving forces behind this activity is that a lot of students identify any **U** or **V** shaped curve as a parabola. It is important that students realise that a parabola has a specific set of properties including a focal point and directrix, and can be constructed geometrically as well as algebraically. Some students appreciate the elegance of the equation $y = x^2$ in expressing such characteristics. The activity also links geometric transformations with the algebraic representations that can produce the same set of transformations.

Example 2 – the discriminant

The discriminant of a quadratic polynomial provides information about its roots. In the case of a quadratic $ax^2 + bx + c$ the discriminant is $b^2 - 4ac$. This activity connects the relatively abstract concept (for students) of the roots of a quadratic with the visual and graphical environment as related to the *x* intercepts.

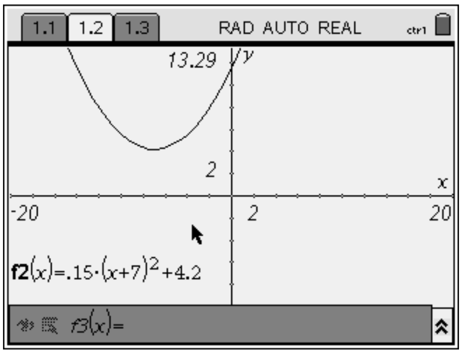
The simplest way to conduct this activity is to provide a specific set of instructions for each student. These instructions fall into two categories:

- draw a parabola that crosses the *x* – axis in two places
- draw a parabola that will never cross or touch the *x* – axis.

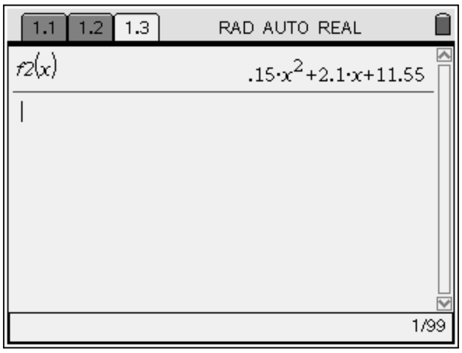
Students graph the function: $y = x^2$

By dragging the vertex, the parabola can be freely translated. Dragging points on the curve other than those near the vertex dilates the graph.

Note: Students are advised that the parabola extends beyond what is displayed on the screen.



If a calculator page is added to the document, the algebraic form of the parabola can be recalled. However, the format is not the same as that displayed on the graphing page; this subtly introduces a different representation of the algebraic form of the parabola.



Students are asked to identify the parameters: *a*, *b* and *c* and substitute these into the rule:

$b^2 - 4ac$

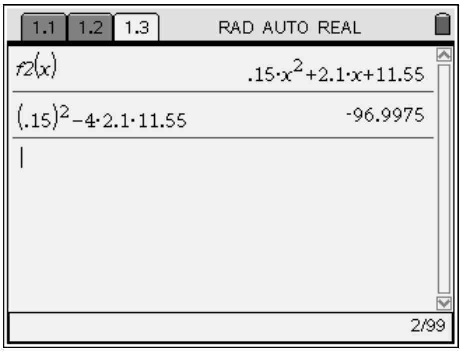
Students sketch the parabola on a piece of paper and write down the value of their discriminant.

Next, students group themselves into two categories:

- discriminant greater than zero
- discriminant less than zero.

A question is posed to the groups:

- What information does the discriminant provide about the graph?
- What would the graph look like if the disciminant is equal to zero?



Students summarise their findings from this investigation. These findings are later attached to factorising quadratics and the third algebraic representation of a quadratic equation. Further questions include:

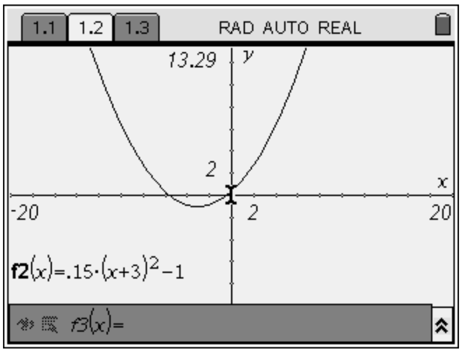
- when the equation is of the form: $y = a(x-h)^2 + k$, how could you tell if the discriminant would be greater than, less than or equal to zero?
- when the equation is of the form: $y = a(x-m)(x-n)$, how could you tell if the discriminant would be greater than, less than or equal to zero?

Example 3 – transformations

In the past, transformations of parabolas have generally relied upon giving the students an appropriate equation and asking them to explore each of the parameters. Without such a *special* equation, this type of investigation is difficult, almost like a secret password. This new platform, TI-*n*spire CAS allows students to explore graphical and the associated algebraic representation without prior knowledge of this *special* representation.

Using a graphs and geometry page, students graph $y = x^2$ and drag the graph around the screen. Ask students to observe carefully the effect translating/dilating the graph has on the algebraic representation.

Note: The *reason* why this representation works the way it does is addressed in a different environment.



Additional questions for students in this activity include:

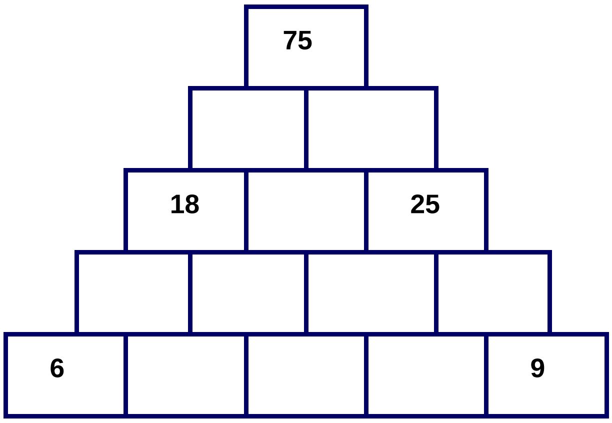
- Can all parabolas be represented in this format?
- Can you tell from the equation (without seeing the graphical representation) whether the graphical representation passes through the *x* – axis?

Year 10 program
Systems of simultaneous linear equations

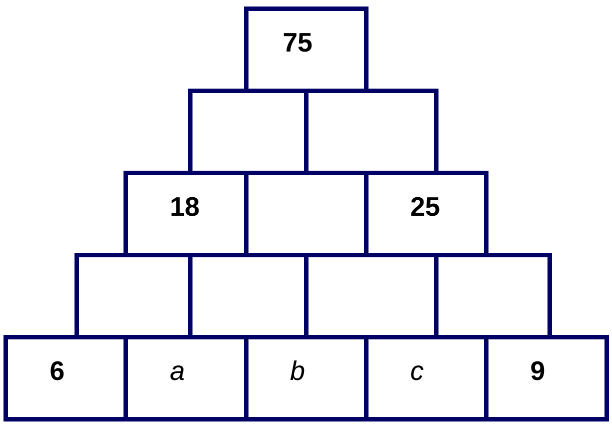
Timeline Semester 2	Level 6 Year 10 Topic	VELS Learning focus	CAS implementation
Weeks 1–6	**Systems of equations** • construct algebraic expressions from a variety of situations including practical problems • interpret equations using natural language • solve sets of equations using technology and simple cases by hand.	*Measurement* • rearrange and simplify algebraic expressions involving real variables *Working mathematically* • use generalisations and arguments in natural language and symbolic form • use technology to carry out symbolic manipulation	Number pyramids Example 1

Example 1 – number pyramids

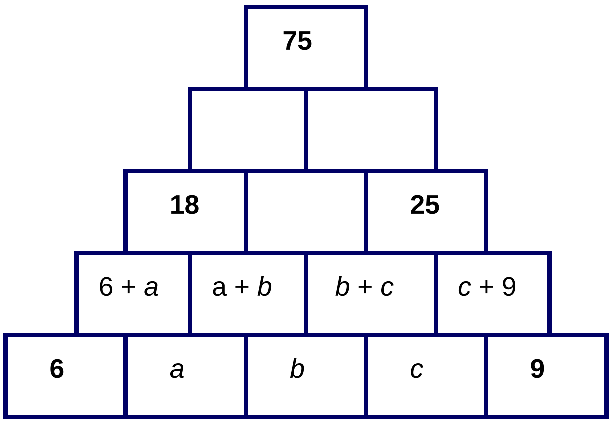
Number pyramids have the property that each brick is equal to the sum of the two bricks upon which it is resting. The number pyramid shown here has a unique solution. However, it is not obvious where to start, as the numbers on the blocks do not provide immediate information about what numbers need to appear on the *blank* blocks. Students are given one more piece of information: *all the numbers on this pyramid are whole number.*



Some students will begin with a *guess and check* process. Providing students with a much bigger pyramid makes this problem a lot more challenging. An algebraic process is soon introduced:



A pertinent question at this point is: ‘Is it necessary to introduce any more variables?’



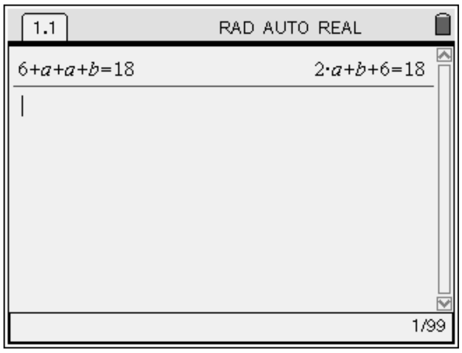
From this point students are required to formulate algebraic expressions for the next row of the pyramid:

$6 + a + a + b = 18$

If students enter this equation on the calculator only a minor simplification is performed by the calculator. Students that write their equations as:

$2a + b = 12$

have preformed some of their own simplifications.

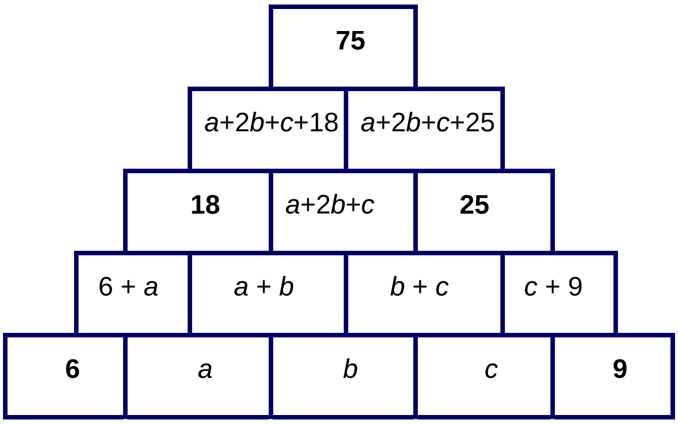


Recall the initial information: ‘All numbers on this pyramid are whole numbers’.

The equation: $2a + b = 12$, provides information about *b*. Students are required to consider values for *b*. Student conclusions extend to:

- ‘*b* must be an even number since 2*a* will be an even number, therefore *b* will be even’
- ‘*b* cannot be bigger than 10’
- ‘*a* cannot be bigger than 5’.

Students can now complete the remainder of the table (algebraically).



Once the blocks have been completed (algebraically) a new set of equations can be formed:

$2a + b = 12$

$b + 2c = 16$

$a + 2b + c = 16$

Discussion can take place about instances where the coefficient of a variable is 1 or 0 in an equation. Each of these linear equations has been formulated with respect to three variables, *a*, *b* and *c*. In the equation $2a + b = 12$, the coefficient of *a* is 2, the coefficient of *b* is 1, and the coefficient of *c* is 0.

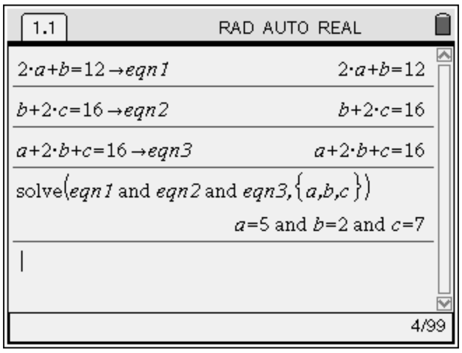
The calculator can be used to solve and check the validity of the set of simultaneous linear equations established in the pyramid. They can also be checked against the *expectations* developed from the earlier equations.

$\{2a + b = 12, b + 2c = 16, a + 2b + c = 16\}$

These equations can be stored such as:

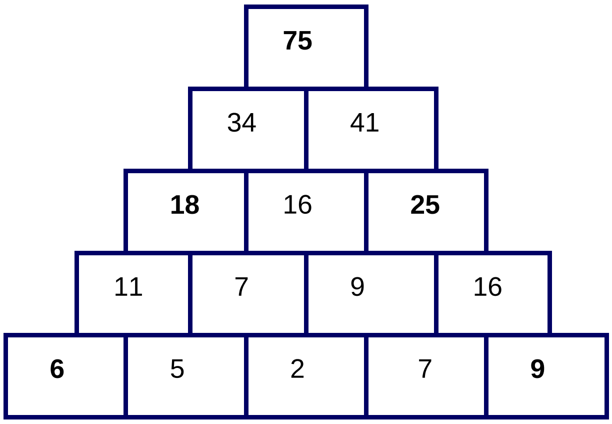
$2a + b = 12$ store in *eqn*1

This makes solving the equations easy (as shown).



Conclusions drawn about $2a + b = 12$ included that *b* must be even; the result agrees with this deduction.

Another conclusion drawn from this equation was that *a* would be no more than 5; this too, is true.



The results found for *a*, *b* and *c* fit the pyramid. After some practice, students may start to draw generalisations from these pyramids:

What is the minimum number of blocks that must contain a number in order for a unique solution to exist in any pyramid problem?

Pyramid problems are a simplified concept for more significant puzzles such as Sudoku puzzles. Here a minimum set of conditions exist in order for a unique solution to exist.