Victorian Essential Learning Standards

CAS Technology Mathematics at Level 6

Part 5: Case Studies – CAS in Practice

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Case studies – the use of Computer Algebra System (CAS) in practice in three schools

Following is one of three sets of course outline materials and related activities developed and used by teachers who have worked with CAS as part of the Victorian Curriculum and Assessment Authority's (VCAA's) Mathematical Methods (CAS) pilot program, and who implemented Mathematical Methods (CAS) Units 1 to 4 in 2006. The course outlines illustrate a variety of approaches, with different emphases, and use of documentation.

Case study B

School

Lavalla Catholic College – catholic, co-educational, regional, Year 7 to Year 12

Curriculum structure

Year 9

Lavalla Catholic College has ten Year 9 classes. There are eight classes at the St Paul's Campus in Traralgon and two classes at the Presentation Campus in Newborough. The Year 9 students are enrolled in the following core subjects: English, Mathematics, Studies of Society and Environment, Science (SOSE), Religious Education, Physical Education, Enterprise Projects, Career Education and Sport and Recreation Activities. Electives are offered including Languages Other Than English (LOTE), Arts, Technology and Music. All Year 9 students participate in a three-day camp at Mt Tamboritha which has an outdoor focus.

Mathematics at Year 9 is timetabled for four 50-minute periods per week. Applied Mathematics is offered in addition to the eight mathematics classes on the St Paul's Campus, these classes are smaller student numbers and are supported by additional needs staff.

Year 10

Lavalla Catholic College has ten Year 10 Mathematics classes at the Kildare Campus in Traralgon. There are two enhancement classes and one Applied Mathematics class. These groups assist students in following path ways that lead into Further Mathematics, Mathematical Methods and Specialist Mathematics. Students also may take up Foundation Mathematics through Victorian Certificate of Applied Learning (VCAL).

Mathematics at Year 10 is timetabled for four 50-minute periods per week. This continues for the whole year except for the examination blocks at the end of both Semesters 1 and 2. The following program is based on two 18-week semesters.

CAS technology used

Mathematica – computer software, site license. Students have access to computer laboratories for one or several lesson/s per week for investigative projects (*Working mathematically* dimension). Students can purchase and install *Mathematica* on their home computers with a three year license available from the school, which would enable them to continue using *Mathematica* in Year 11 and 12 Mathematics choices.

In Year 11 and Year 12 students are timetabled in computer laboratories for Mathematical Methods (CAS) and Specialist Mathematics.

Overview Level 6 Year 9

Approximate time	Skills	Dimensions
	Semester 1	
	Number Systems	**Number**

2 weeks
(weeks 1–2)
 	
2 weeks
(weeks 3–4)
 	
1 week
(week 5)
	The real number system
	• Classifying numbers: Definitions of Rational and Irrational numbers, Integers, Natural Numbers, Surds and Transcendental Numbers.
	Integers
	• Working with integers: order of operations.
	• Rules for divisibility
	• Finding prime factors and the highest common factor for two numbers.
	• Using the Euclidean Algorithm to find the greatest common divisor of two numbers.
	Rational Numbers
	• Working with Fractions:
	  o equivalent fractions
	  o addition, subtraction and multiplication
	Assessment: Technology free skills test
	• Working with decimals numbers
	  o converting fractions to decimals
	  o expressing recurring decimals as fractions
	  o percentages and their applications
	  o ratios
	Irrational Numbers
	• Working with surds – simplifying, adding, subtracting, multiplying and dividing.
	• Estimating the size of surds
	Scientific notation
	• Writing large numbers in standard form
	• Perform calculations involving very large or very small numbers using scientific notation
	Assessment: Test (CAS can be used)
		At Level 6, students comprehend the set of real numbers containing natural, integer, rational and irrational numbers. They represent rational numbers in both fractional and decimal (terminating and infinite recurring) forms (for example, $1\frac{4}{25} = 1.16, 0.\overline{47} = \frac{47}{99}$).
		They comprehend that irrational numbers have an infinite non-terminating decimal form. They specify decimal rational approximations for square roots of primes, rational numbers that are not perfect squares …
		
		Students use the Euclidean division algorithm to find the greatest common divisor (highest common factor) of two natural numbers (for example, the greatest common divisor of 1071 *and* 1029 *is* 21 *since* $1071 = 1029 \times 1 + 42$, $1029 = 42 \times 24 + 21$ and $42 = 21 \times 2 + 0$).
		
		Students carry out arithmetic computations involving natural numbers, integers and finite decimals using mental and/or written algorithms (one- or two-digit divisors in the case of division). They perform computations involving very large or very small numbers in scientific notation
		(*for example,* $0.0045 \times 0.000028 = 4.5 \times 10^{-3} \times 2.8 \times 10^{-5} = 1.26 \times 10^{-7}$).
		
		They carry out exact arithmetic computations involving fractions and irrational numbers such as square roots
		(*for example,* $\sqrt{18} = 3\sqrt{2}, \sqrt{(\frac{3}{2})} = \frac{\sqrt{6}}{2}$)

	Measurement	
2 weeks (weeks 6–7)	• Metric units and conversion of measurement units • Errors and approximation • Perimeters of different shapes, circumferences of a circles and arc lengths • Areas of composite figures • Total surface area • Volume and capacity Activity: Estimating area of composite figures	*At Level 6, students estimate and measure length, area, surface area, mass, volume, capacity … They select and use appropriate units, converting between units as required … Students decide on acceptable or tolerable levels of error in a given situation. They interpret and use mensuration formulas for calculating the perimeter, surface area and volume of familiar two- and three-dimensional shapes and simple composites of these shapes …*
	Linear Relationship	**Structure**
2 weeks (week 8–9)	***Linear equations:*** • Solving basic linear equations (review) • Solving two- and three-steps linear equations • Solving more complex equations where unknown appears more than once • Transposing formulae (*Mathematica* can be used to verify answers) • Applications/analysis problems/modelling a range of contexts • Solving linear equations using *Mathematica* Assignment	*They solve equations of the form* $f(x) = k$*, where k is a real constant … using algebraic, numerical* [*systematic guess, check and refine or bisection*] *and graphical methods.*
1 week (week 10)	***Linear inequalities:*** • Exploring inequalities (set notation) • Solving linear inequalities • applications/analysis problems/ modelling a range of contexts Assessment: Topic test	

2 weeks (weeks 11–12)	***Linear graphs*** Plotting linear graphs • Plotting points on the Cartesian plain (review) • Plotting linear graphs by table (recognition of independent and dependent variables, domain and range) • Plotting linear graphs using *x*- and *y*- intercepts • Gradient of a straight line • Horizontal and vertical lines • Plotting linear graph using the gradient and y-intercept • Plotting linear equations using *Mathematica* • Finding the equation of a line • Applications/analysis problems/ modelling a range of contexts Activities: Graph activities using technology Assignment: Birthday party	*Students identify and represent linear… functions by table, rule and graph … with consideration of independent and dependent variables, domain and range.* *They recognise and explain the roles of the relevant constants in the relationships* $f(x) = ax + c$*, with reference to gradient and* y *axis intercept …*
2 weeks (weeks 13–14)	***Simultaneous equations:*** • Solving simultaneous equations using graphs • Solving simultaneous equations using the substitution and elimination methods; verifying solutions using tables • Applying simultaneous equations to solve problems with two unknowns (*Mathematica* can be used) • Identifying the condition for simultaneous equations with no solutions Assessment task	*They solve … simultaneous linear equations in two variables* (*for example,* $\{2x - 3y = -4$ and $5x + 6y = 27\}$) *using algebraic, numerical* (*systematic guess, check and refine or bisection*) *and graphical methods.*
	Index form	**Structure**
2 weeks (week 15–16)	• Writing expressions with common factors in index form. • Applying index laws (including the use of negative powers) • Simplifying expressions Assessment: Technology free skills test	*Students apply the algebraic properties* (*closure, associative, commutative, identity, inverse and distributive*) *to computation with number, to rearrange formulas, rearrange and simplify algebraic expressions involving real variables. They verify the equivalence or otherwise of algebraic expressions* (*linear, square, cube, exponent, and reciprocal*)*, for example,* $4x - 8 = 2(2x - 4) = 4(x - 2)$; $(2a - 3)^2 = 4a^2 - 12a + 9$; $(3w)^3 = 27w^3$; $\frac{(x^3y)}{xy^2} = x^2y^{-1}$; $\frac{4}{xy} = \frac{2}{x} \times \frac{2}{y}$.

	Pythagoras	Measurement, chance and data
2 weeks (weeks 17–18)	• Solving a right-angle triangle using pythagoras theorem in exact and approximate form • Finding side length of composite shapes • Using pythagoras theorem in the Cartesian plane • Pythagoras in three dimensions • Applications Problem-solving task	*Students use pythagoras theorem … to obtain lengths of sides, angles and the area of right-angled triangles.*
Semester 2		
	Geometry	**Space**
1 week (week 1)	• Properties of Polygons: o Triangles ➢ basic types of triangles ➢ interior angles ➢ exterior angles in a triangles o Quadrilaterals (*Exploring approach*) ➢ basic types of quadrilaterals ➢ angles in quadrilaterals ➢ cyclic quadrilaterals	*… students represent two- and three-dimensional shapes using lines, curves, polygons and circles.*
1 week (week 2)	• Constructions • Scale drawings • Enlarging or reducing. Scale factor o side length, o area, o volume o angle measure Assignment: Building Plans	*They determine the effect of changing the scale of one characteristic of two- and three-dimensional shapes* (*for example, side length, area, volume and angle measure*) *on related characteristics.*
1 week (week 3)	• Congruent triangles	*Students use the conditions for shapes to be congruent or similar.*
1 week (week 4)	• Properties of similar triangles • Undirected graphs and networks • Planer graphs and Euler’s formula Activity: Fractals	*Students describe and use the connections between objects/location/events according to defined relationships* (*networks*).

	Ratios and Rates	**Number**

1 week
(week 5)
 	
2 weeks
(weeks 6–7)
	• Fractions and ratios
	• Ratios and quantities
	• Rates of change
	• Rates and units
	• Comparing using ratios, percentage or rates
	Applications: Solar system or Heart rates and health
	Percentage and Ratios
	• Ratios
	• Working with percentages
	• Percentage change
	• Application of percentages
	  o Discount
	  o Profit and loss
	  o Commission
	  o Simple interest
	Investigation project: Income taxation
	Assessment: Topic Test
		Students carry out arithmetic computations involving natural numbers, integers and finite decimals using mental and/or written algorithms (*one- or two-digit divisors in the case of division*).
		
		They use appropriate estimates to evaluate the reasonableness of the results of calculations involving rational and irrational numbers, and the decimal approximations for them. They carry out computations to a required accuracy in terms of decimal places and/or significant figures.

	Quadratic relationship	**Structure**

2 weeks
(weeks 8–9)
	Expanding and Factorising
	• expanding
	  o expanding two factors of the type $a(b + c)$
	  o expanding two factors of the type $(a + b)\ (c + d)$
	  o perfect squares and the difference of two squares
	• factorising
	  o using common factors
	  o by grouping
	  o using the difference of two squares and the perfect square rules
	  o quadratic trinomials
	Assessment: Technology free skills test
		Students apply the algebraic properties … to simplify algebraic expressions involving real variables. They verify the equivalence or otherwise of algebraic expressions (*for example,* $4x - 8 = 2(2x - 4) = 4(x - 2)$*;* $(2a - 3)^2 = 4a^2 - 12a + 9$) …

3 weeks (weeks 10–12)	***Quadratic Graphs and Equations*** • Plotting parabolas • Parabolas and transformations o $y = ax^2$, where $a > 0$ and $a < 0$ o $y = ax^2 + k$ o $y = a(x - h)^2 + k$ • Solving quadratic equations using the null factor law and graphical methods. Assignment: Exploring families of curves (CAS activity)	*Students identify and represent … quadratic … functions by table, rule and graph … with consideration of independent and dependent variables, domain and range.* *They recognise and explain the roles of the relevant constants in the relationships …* $f(x) = a(x + b)^2 + c$ *…* *They solve equations of the form* $f(x) = k$*, where k is a real constant (for example,* $x(x + 5) = 100$*) … using algebraic, numerical (systematic guess, check and refine or bisection) and graphical methods.*
	Data	**Measurement, chance and data**
2 weeks (weeks 13–14) 1 week (week 15)	• Data types, their representation and interpretation • Scatter plots and the line of best fit Applications task: Designing and analysing a survey	*Students comprehend the difference between a population and a sample. They generate data using surveys, experiments and sampling procedures. They calculate summary statistics for centrality (mode, median and mean), spread (box plot, inter-quartile range, outliers) and association (by-eye estimation of the line of best fit from a scatter plot). They distinguish informally between association and causal relationship in bi-variate data, and make predictions based on an estimated line of best fit for scatter-plot data with strong association between two variables.*
	Trigonometry	**Measurement, chance and data**
2 weeks (weeks 16–17) 1 week (week 18)	• Exploring trigonometric ratios • Solving right angle triangles o finding lengths and o angles • Application of trigonometric ratios o angles of elevation and depression o bearings o two- and three-dimensional problems Assessment: Topic test	*Students use pythagoras theorem and trigonometric ratios (sine, cosine and tangent) to obtain lengths of sides, angles and the area of right-angled triangles.*

Overview Level 6 Year 10		
Approximate time	Skills	Dimensions
Semester 1		
	Number Systems	**Number/Structure**
2 weeks (weeks 1–2)	***The real number system*** • Revision of Number systems • Revise working with integers and rational numbers • Introduction to Factorials • Working with Ratios Activity: The Golden Ratio ***Irrational Numbers*** *Surds* • Revision addition, subtraction and multiplication of surds • Rationalising the denominator *Matrices* • The system of 2 × 2 matrices: • addition, subtraction and multiplication of 2 × 2 matrices • multiplication by a scalar Assessment: Technology free skills test	*At Level 6, students comprehend the set of real numbers containing natural, integer, rational and irrational numbers. They represent rational numbers in both fractional and decimal (terminating and infinite recurring) forms (for example,* $1\frac{4}{25} = 1.16, 0.\overline{47} = \frac{47}{99}$). *They comprehend that irrational numbers have an infinite non-terminating decimal form. They specify decimal rational approximations for square roots of primes, rational numbers that are not perfect squares, the golden ratio* φ*, and simple fractions of* π *correct to a required decimal place accuracy.* *Students use the Euclidean division algorithm to find the greatest common divisor (highest common factor) of two natural numbers (for example, the greatest common divisor of* 1071 *and* 1029 *is* 21 *since* $1071 = 1029 \times 1 + 42$, $1029 = 42 \times 24 + 21$ *and* $42 = 21 \times 2 + 0$). *They carry out exact arithmetic computations involving fractions and irrational numbers such as square roots (for example,* $\sqrt{18} = 3\sqrt{2}, \sqrt{(\frac{3}{2})} = \frac{\sqrt{6}}{2}$) … *They use appropriate estimates to evaluate the reasonableness of the results of calculations involving rational and irrational numbers, and the decimal approximations for them. They carry out computations to a required accuracy in terms of decimal places and/or significant figures.*

	Indices	**Structure**
2 weeks (weeks 3–4) 1 week (week 5)	• Using index laws to simplify algebraic expressions • Working with fractional powers • Graphing simple exponential curves • Exploring a family of curves $f(x) = ca^x$ • The 'tower of Hanoi' activity ***Logarithms*** • Introduction to logarithms • Applying laws of logarithms Assessment: Technology free skills test	*They verify the equivalence or otherwise of algebraic expressions (…, exponent …) for example* $(3w)^3 = 27w^3$; $\frac{(x^3y)}{xy^2} = x^2y^{-1}$; $\frac{4}{xy} = \frac{2}{x} \times \frac{2}{y}$ …) *Students identify and represent … exponential functions by table, rule and graph (all four quadrants of the Cartesian coordinate system) with consideration of independent and dependent variables, domain and range. They distinguish between linear, quadratic and exponential functions by testing for constant first difference, constant second difference or constant ratio between consecutive terms (for example, to distinguish between the functions described by the sets of ordered pairs* {(1, 2), (2, 4), (3, 8), (4, 16) …}). *They use and interpret the functions in modelling a range of contexts.* *They recognise and explain the roles of the relevant constants in the relationships* $f(x) = ca^x$ …
	Business maths	**Number**
2 weeks (weeks 6–7)	• Budgeting • Reviewing percentages • Profit and loss • Commission • Simple interest • Compound interest • Investing money • Exploring reducing balance loans Investigative activity: Council Rates or Water Rates Assessment: Topic test	*Students carry out arithmetic computations involving natural numbers, integers and finite decimals using mental and/or written algorithms (one- or two-digit divisors in the case of division). They perform computations involving very large or very small numbers in scientific notation* (*for example,* $0.0045 \times 0.000028 = 4.5 \times 10^{-3} \times 2.8 \times 10^{-5} = 1.26 \times 10^{-7}$).

	Linear relationship	**Structure**
1 week (week 8)	***Linear equations*** Revision of linear equations • solving linear equations by performing inverse operations • solving linear equations where the unknown appears more than once • applications and analysis problems • literal equations	*Students solve equations of the form $f(x) = k$, where k is a real constant … using algebraic, numerical (systematic guess, check and refine or bisection) and graphical methods.*
2 weeks (weeks 9–10)	***Linear graphs*** • Represent a linear function by a table of values, a graph, and by a rule • Describe and specify the independent variable and its domain, and the dependent variable and its range • Plotting linear graphs using x- and y- intercepts • Plotting linear graph using the gradient and y-intercept • Investigation using *Mathematica* • Solving worded problems Assessment: Topic test	*Students identify and represent linear … functions by table, rule and graph (all four quadrants of the Cartesian coordinate system) with consideration of independent and dependent variables, domain and range.* *They recognise and explain the roles of the relevant constants in the relationships $f(x) = ax + c$, with reference to gradient and y axis intercept.*
2 weeks (weeks 11–12)	***Simultaneous equations*** • solving simultaneous equations with two variables using algebraic, numerical and graphical methods • solving simultaneous equations using matrices • applying simultaneous equations to solve problems with two unknowns (*Mathematica* can be used) Assignment: Optimisation problem *Comparing Tour Companies*	*Students solve … simultaneous linear equations in two variables, for example $\{2x - 3y = -4$ and $5x + 6y = 27\}$, using algebraic, numerical (systematic guess, check and refine or bisection) and graphical methods.*

	Chance	**Measurement, chance and data/Structure**

3 weeks (weeks 13–15)	• Estimation probability based on data
	  o experiments
	  o surveys
	  o samples
	  o simulations
	• Theoretical probability
	• Venn diagrams and additional Law of probability
	• Karnaugh maps
	• Probability tree diagrams
	• Mutually exclusive events
	• Independent events
	• Conditional probability
	• Applications
	Assessment: From London to Paris on the Chunnel train
	Assessment: Topic test
		Students estimate probabilities based on data (*experiments, surveys, samples, simulations*) *and assign and justify subjective probabilities in familiar situations. They list event spaces* (*for combinations of up to three events*) *by lists, grids, tree diagrams, venn diagrams and karnaugh maps* (*two-way tables*)*. They calculate probabilities for complementary, mutually exclusive, and compound events* (*defined using* and, or *and* not)*. They classify events as dependent or independent.*
		Student express relations between sets using membership, $\in$, *complement,* $'$ *, intersection ,* $\cap$*, union,* $\cup$*, and subset,* $\subseteq$ *, for up to three sets. They represent a universal set as the disjoint union of intersections of up to three sets and their complements, and illustrate this using a tree diagram, venn diagram or karnaugh map.*

	Trigonometry and circular (trigonometric) functions	**Measurement, chance and data/Structure**

3 weeks (weeks 16–18)	• Trigonometric ratios
	• Solving right angle triangles
	  o finding lengths and
	  o angles
	• Application of trigonometric ratios
	  o angles of elevation and depression
	  o bearings
	  o two- and three-dimensional problems
	• Unit circle
	• Exploring circular (trigonometric) functions
	Activity: Exploring the symmetry properties of the unit circle
	Assessment: Topic test
		Students use pythagoras theorem and trigonometric ratios (*sine, cosine and tangent*) *to obtain lengths of sides, angles and the area of right-angled triangles.*
		They use degrees and radians as units of measurement for angles and convert between units of measurement as appropriate.

Semester 2		
	Measurements	**Measurement, chance and data**
2 weeks (weeks 1–2)	• Pythagoras in two- and three- dimensions • Activity: Proving area formulae • Estimate, measure and calculate perimeter and area of composite shapes • Exploring the surface area and volume of o a cylinder o a cone • Total surface area of a solid (including a composite solid) • Volume of prisms, regular solids and sphere • Capacity • Density, concentration and speed • Accuracy of measuring devices • Errors and tolerance • Problem solving Design a housing estate Assessment: Packing spheres	*At Level 6, students estimate and measure length, area, surface area, mass, volume, capacity and angle. They select and use appropriate units, converting between units as required. They calculate constant rates such as the density of substances* (*that is, mass in relation to volume*)*, concentration of fluids, average speed and pollution levels in the atmosphere. Students decide on acceptable or tolerable levels of error in a given situation. They interpret and use mensuration formulas for calculating the perimeter, surface area and volume of familiar two- and three-dimensional shapes and simple composites of these shapes.*
	Quadratic Relationship	**Structure**
2 weeks (weeks 3–4)	***Expanding and factorising*** • expanding o two factors o perfect squares and the difference of two squares o three factors • factorising o using common factors o by grouping o using the difference of two squares and the perfect square rules o quadratic trinomials o by completing the square • Problem solving: Designing a garden path Assessment: Technology free skills test	

4 weeks
(week 5 –8)
	Quadratic Graphs and Equations
	• solving quadratic equations
	  o using the null factor law
	  o by completing the square
	  o using quadratic formula
	  o graphically
	• sketching parabolas
	  o in turning point form $y = a(x - h)^2$
	  o in the form $y = ax^2 + bx + c$
	  o parabola transformations
	• Problem solving:
	  o Design a water feature;
	  o Playing golf
	Assignment: Difference equations
		Students identify and represent … quadratic … functions by table, rule and graph (*all four quadrants of the Cartesian coordinate system*) *with consideration of independent and dependent variables, domain and range. They distinguish* [*between linear, quadratic and exponential*] *functions by testing for constant first difference, constant second difference or constant ratio between consecutive terms* (*for example, to distinguish between the functions described by the sets of ordered pairs* {(1, 2), (2, 4), (3, 6), (4, 8) …}; {(1, 2), (2, 4), (3, 8), (4, 14) …}; and {(1, 2), (2, 4), (3, 8), (4, 16) …}). *They use and interpret the functions in modelling a range of contexts.*
		
		They recognise and explain the roles of the relevant constants in the relationships $f(x) = a(x + b)^2 + c$ *…*
		
		They solve equations of the form $f(x) = k$, *where k is a real constant* (*for example,* $x(x + 5) = 100$) *using algebraic, numerical* (*systematic guess, check and refine or bisection*) *and graphical methods.*

	Data	**Measurement, chance and data**

3 weeks
(weeks 9–11)
	• Univariate and bivariate data, their representation and interpretation
	  o Scatter plots and line of best fit
	  o Representation time series
	• Application using CAS
	Application task: Caring for environment
		Students comprehend the difference between a population and a sample. They generate data using surveys, experiments and sampling procedures. They calculate summary statistics for centrality (*mode, median and mean*), *spread* (*box plot, inter-quartile range, outliers*) *and association* (*by-eye estimation of the line of best fit from a scatter plot*). *They distinguish informally between association and causal relationship in bi-variate data, and make predictions based on an estimated line of best fit for scatter-plot data with strong association between two variables.*

	Geometry	**Space**
1 week (week 12) 1 week (week 13) 1 week (week 14) 2 weeks (weeks 15–16)	• Circles geometry o the features of circles o associated angle properties o arc length • Three dimensions o isometric drawing o elevations and plans o pythagoras in three dimensions (including review of pythagoras in two dimensions) • Constructions • Transformation o translation o reflection o rotation o dilation • Congruent shapes (two- and three-dimensional shapes) • Similar figures/similar triangles. • Scale factor o side length o area o volume o angle measure Investigation: Great circles	*At Level 6, students represent two- and three-dimensional shapes using lines, curves, polygons and circles. They make representations using perspective, isometric drawings, nets and computer-generated images. They recognise and describe boundaries, surfaces and interiors of common plane and three-dimensional shapes, including cylinders, spheres, cones, prisms and polyhedra. They recognise the features of circles (centre, radius, diameter, chord, arc, semi-circle, circumference, segment, sector and tangent) and use associated angle properties.* *Students explore the properties of spheres.* *Students use the conditions for shapes to be congruent or similar. They apply isometric and similarity transformations of geometric shapes in the plane. They identify points that are invariant under a given transformation (for example, the point (2, 0) is invariant under reflection in the x-axis, so the x axis intercept of the graph of* $y = 2x - 4$ *is also invariant under this transformation). They determine the effect of changing the scale of one characteristic of two- and three-dimensional shapes (for example, side length, and area, volume and angle measure) on related characteristics.* *They use latitude and longitude to locate places on the Earth's surface and measure distances between places using great circles.*
2 weeks (weeks 17–18)	***Families of functions (overview)*** • Linear functions • Quadratic functions • Cubic functions • Exponential functions • Hyperbolas • Difference equations • Transformation	*They distinguish between these types of functions by testing for constant first difference, constant second difference or constant ratio between consecutive terms. They recognise and explain the roles of the relevant constants in the relationships* $f(x) = ax + c$, *with reference to gradient and y axis intercept,* $f(x) = a(x + b)^2 + c$ *and* $f(x) = ca^x$.

Rational Numbers

Number

At Level 6, students carry out exact arithmetic computations involving fractions and irrational numbers such as square roots (for example, $\sqrt{18} = 3\sqrt{2}, \sqrt{(\frac{3}{2})} = (\frac{\sqrt{6}}{2})$ *) … They use appropriate estimates to evaluate the reasonableness of the results of calculations involving rational and irrational numbers, and the decimal approximations for them.*

Approximating surds using iteration

The process of iteration can be used to calculate the approximate value of a square root.
$\sqrt{a}$ can be found by using the formula: $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$.

Using CAS, students can create a table of values which will enable them to observe convergence.

Exploring addition subtraction, multiplication and division of surds

CAS is a useful tool for exploring the properties of surds. Often the answers given by CAS are given in a form which requires students to recognise equivalent expressions:

Example:

a. **Expand** $[\sqrt{3}(\sqrt{5}+\sqrt{6}]$
$3\sqrt{2}+\sqrt{15}$

b. **Expand** $[\sqrt{5}(\sqrt{15}+\sqrt{30})]$
$5\sqrt{3}+5\sqrt{6}$

c. **Together** $[\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{3}}]$
$\frac{1}{15}(10\sqrt{3}+3\sqrt{5})$

Similarly, CAS can be used as a teaching tool for helping students understand the operations involving fractions.

Example:

Using CAS find $\frac{2}{a}+\frac{3}{b}$.

Together $[\boldsymbol{\frac{2}{a}+\frac{3}{b}}]$

$$\frac{3a+2b}{ab}$$

Can you explain how we get this answer?

Similarly:

a. $\boldsymbol{\frac{2}{a}\times\frac{3}{b}}$

$$\frac{6}{ab}$$

b. $\boldsymbol{\frac{2}{a}\div\frac{3}{b}}$

$$\frac{2b}{3a}$$

Linear relationship

Structure

They solve equations of the form $f(x) = k$, where k is a real constant and simultaneous linear equations in two variables (for example, $\{2x - 3y = -4$ and $5x + 6y = 27\}$) using algebraic, numerical (systematic guess, check and refine or bisection) and graphical methods.

Students can solve some linear equation with one unknown either by rule, using the table (numerical method) or graphically.

Example 1

Solve the following equation.

$$2x + 7 = 17$$

Solution by rule

$$2x + 7 - 7 = 17 - 7$$
$$2x = 10$$
$$\frac{2x}{2} = \frac{10}{2}$$
$$x = 5$$

Solution using a table

Students can define a linear function $f(x)$ and then recognise the connection between the input and the resulting output value of $f(x)$. This can be observed by the students using a table.

In [1]: = **f[x_]: = 2x + 7**

In [2]: = **Table [{x, f[x]}, {x, –3, 6}] / / TableForm**

Out [2]/Table Form =

–3	1
–2	3
–1	5
0	7
1	9
2	11
3	13
4	15
5	17
6	19

From this table students can note that $2 \times 5 + 7 = 17$, so the solution of the equation is $x = 5$.

The use of function notation $f(x) = 2x + 7$ gives the opportunity to easily check the solution: $f(5) = 17$.

```
f[x_] : = 2x + 7
f[5]
17
```

Example 2

Solve the following equation

$$3(x-2) = 2(x+1)$$

Solution by rule

$3(x – 2) = 2(x + 1)$
$3x – 6 = 2x + 2$
$3x – 2x = 2 + 6$
$x = 8$

Solution using a table

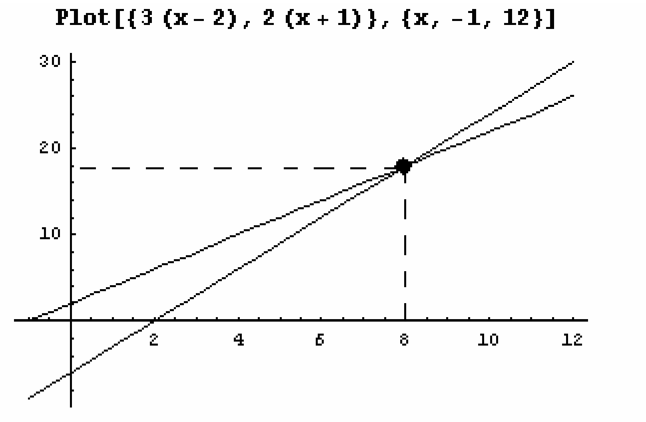
Table [{x, 3(x – 2), 2(x + 1)}, {x, 5, 10}] / / TableForm

5	9	12
6	12	14
7	15	16
8	18	18
9	21	20
10	24	22

From this table students can conclude that $x = 8$, when LHS = RHS of the equation.

Solution using a graph

The students can solve this equation graphically and give a solution as the x coordinate of the point of intersection of a straight line with equation $y = 3(x - 2)$ and a straight line with equation $y = 2(x + 1)$.



To confirm that the point of intersection is $x = 8$ **FindRoot** command can be used:

Find Root [3(x – 2) == 2(x + 1), {x, 8}]

{x → 8.}

therefore $x = 8$

To check their solution students can use **Solve** command:

Solve [3(x – 2) == 2(x + 1), x]

{{x → 8}}

Students' knowledge can be extended to solve simultaneous equations using *algebraic, numerical and graphical methods.*

Example 3

Solve the simultaneous equations.

$$\begin{cases} 2x + 3y = 14 \\ -x + 5y = 6 \end{cases}$$

These simultaneous equations can be solved algebraically by elimination or substitution.

In order to solve these equations graphically, students must rearrange the equations to make *y* the subject.

$$\begin{cases} 3y = 14 - 2x \\ 5y = 6 + x \end{cases}$$

$$\begin{cases} y = \dfrac{14 - 2x}{3} \\ y = \dfrac{6 + x}{5} \end{cases}$$

Using *Mathematica*:

Roots[2 x + 3 y == 14, y]

$$y == -\frac{2}{3}(-7 + x)$$

Roots[-x + 5 y == 6, y]

$$y == \frac{6 + x}{5}$$

These pair of simultaneous equations will graph as two straight lines. The solution of the equations is given by the coordinates of the point of intersection of the lines.
To find x and y coordinate of the point of intersection from the graph:

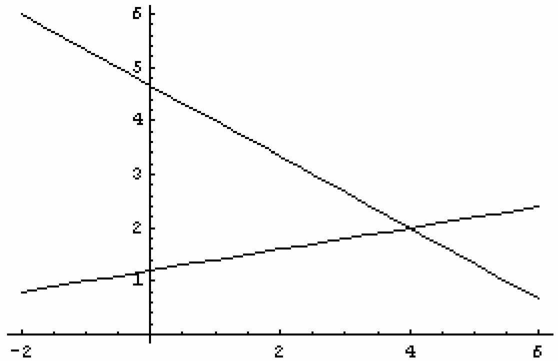
```
FindRoot[(14 - 2 x)/3 == (6 + x)/5, {x, 4}]
```

{x → 4.}

```
Solve[y == (14 - 2 x)/3 /. x -> 4]
```

{{y → 2}}

```
Plot[{(14 - 2 x)/3, (6 + x)/5}, {x, -2, 6}]
```



Therefore, the point of intersection is (4, 2), so the solution of the simultaneous equation is: $x = 4$ and $y = 2$.

Checking:

$$\begin{cases} 2\times 4 + 3\times 2 = 14 \\ -4 + 5\times 2 = 6 \end{cases}$$

Or

```
Solve[{y == (14 - 2 x)/3, y == (6 + x)/5}, {x, y}]
```

{{x → 4, y → 2}}

Using matrices

The simultaneous equations can be represented using matrices:

$$\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 6 \end{bmatrix}$$

```
Solve[(2 3; -1 5).(x; y) == (14; 6), {x, y}]
```

{{x → 4, y → 2}}

Structure/Working Mathematically

They recognise and explain the roles of the relevant constants in the relationships $f(x) = ax + c$, with reference to gradient and y *axis intercept …*
They select and use technology in various combinations to assist in mathematical inquiry, to manipulate and represent data, to analyse functions and carry out symbolic manipulation …

Example 4

Identify the key features of the graph of $y = mx + c$

To investigate how parameters m and c effect the position and shape of a linear graph one of these values has to be fixed and the second parameter will be variable.

1. Students are familiar with graph of $y = x$

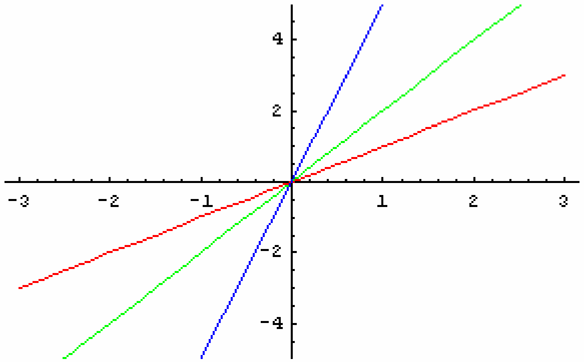
   a. Explore the graphs of
   $y = x + 5$
   $y = x - 3$

   b. Predict the position of the graph
   $y = x + c$

   At the completion of the above activities students will be able to recognise the effect on the graph of changing the value of c, the y-intercept.

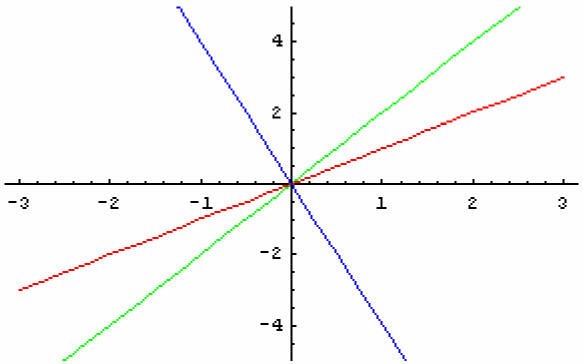
   c. Further explore
   $y = 2x$
   $y = 5x$

```
Plot[{x, 2 x, 5 x}, {x, -3, 3}, PlotRange → {-5, 5},
 PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 1, 0],
   RGBColor[0, 0, 1]}]
```



$y = -4x$

```
Plot[{x, 2 x, -4 x}, {x, -3, 3}, PlotRange → {-5, 5},
 PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 1, 0],
   RGBColor[0, 0, 1]}]
```



d. Predict the position of the graph
$y = mx$

e. Predict the position of the general graph with parameters m and c
$y = mx + c$

f. explain the roles of m and c in the relation $y = mx + c$

From these activities students can recognise the effect on the graph of changing the values of m and c, the gradient and the y-intercept respectively. The following examples may be used for students who are working above level 6 in this dimension.

Example 5

For what value of p will the following simultaneous equations

$$\begin{cases} px + y = 4 \\ 3x + 2y = 5 \end{cases}$$

have no solutions?

Example 6

For what values of p and q will the following simultaneous equations

$$\begin{cases} px + 2y = 4 \\ 2x + 3y = q \end{cases}$$

have

a. no solutions?
b. infinitely many solutions?
c. a unique solution?

Exploring exponential relationships

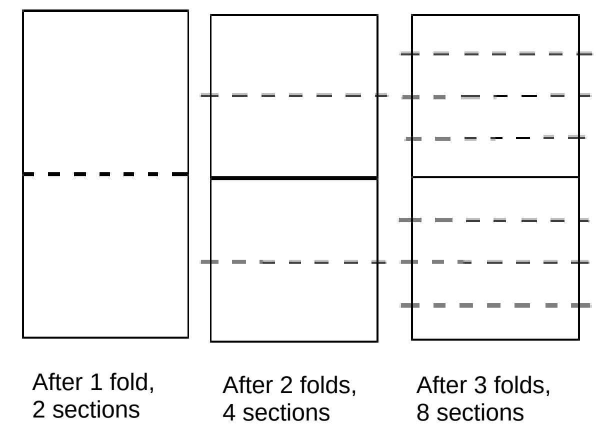
Structure

Students identify and represent … exponential functions by table, rule and graph (all four quadrants of the cartesian coordinate system) with consideration of independent and dependent variables, domain and range …

Example 1

The following problem-solving activity provides a good introduction lesson for exponential functions. Students are introduced to the basic properties of exponential functions through the process of continually halving a piece of paper. On completing this task they will be able to determine the appropriate rule for the function.

Paper tearing
You have a single sheet of A4 paper. When you fold it in half, you have two sections of paper. If you fold the sheet in half again, you will have four sections, and so on.



Copy and complete the table below, which gives the number of sections *S(n)* as a function of the number of folds *n*.

Number of folds *n*	0	1	2	3	4	5	6
Number of sections *S(n)*	1	2	4	8	16	32	64

By observing the table, and thinking about the action of continually folding sheet of paper in half, we can work out a rule for $S(n)$ in terms of n.

Number of sections after 0 folds = $S(0) = 1$
Number of sections after 1 folds = $S(1) = S(0) \times 2 = 1 \times 2 = 1 \times 2^1$
Number of sections after 2 folds = $S(2) = S(1) \times 2 = 1 \times 2 \times 2 = 1 \times 2^2$
Number of sections after 3 folds = $S(3) = S(2) \times 2 = 1 \times 2 \times 2 \times 2 = 1 \times 2^3$

. . . . .

. . . . .

. . . . .

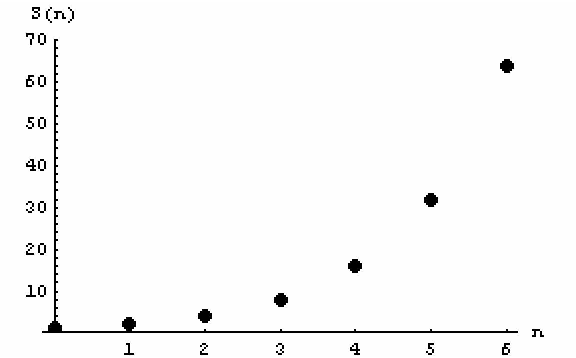
Number of sections after n folds = $S(n) = 1 \times 2 \times 2 \times 2 \times \ldots \times 2 = 1 \times 2^n$

So $\mathbf{S(n) = 1 \times 2^n = 2^n}$

The function $S(n) = 2^n$ is an example of an exponential function. The independent variable n is the power or exponent and the constant 2 is the base. In general, exponential functions have the independent variable in the exponent.

If we substitute $n = 0$ into the rule $S(n) = 2^n$, the rule gives $S(0) = 2^0 = 1$. Since we know that $S(0) = 1$, this means $\mathbf{2^0 = 1}$. In general, if a is not equal to zero we get $\mathbf{a^0 = 1}$.

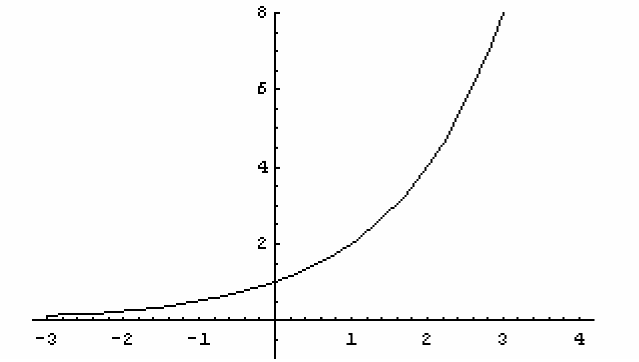
Using the data from the table above, students can plot the graph by hand:



Now we can use the function $S(n)$ to plot the graph and identify the properties of the exponential function:

- the y-intercept (0,1);
- graph passes through the point (1,2);
- Domain: R and Range: R^+;
- the asymptote $y = 0$

```
In[1]:= S[n_] := 2^n

In[2]:= Plot[S[n], {n, -3, 4}, PlotRange → {-1, 8}]
```



Note: Although the function has been plotted as though the number of folds *n* can take any value, in fact only whole number values are possible for *n* in this case.

Questions:

a. From the graph, how many sections would there be after you have made 10 folds?

b. From the graph, how many folds will give 256 sections?

Using *Mathematica* students can check their answers:

a.

```
In[4]:= 2^10

Out[4]= 1024
```

Or

```
In[5]:= S[10]

Out[5]= 1024
```

b.

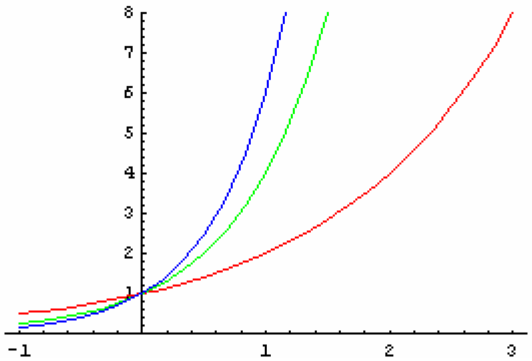
```
In[6]:= Solve[2^n == 256, n]

Out[6]= {{n → 8}}
```

Students extend their knowledge of exponential functions to include exponential functions with different bases and function of the form $y = ca^x$. They recognise that changing the base effects the steepness of the curve and that the inclusion of a coefficient effects the steepness and the y-intercept of the curve.

Example 2

To establish properties of exponential graphs with different bases several graphs can be plotted on the same set of axes.

```
Plot[{2^x, 4^x, 6^x}, {x, -1, 3}, PlotRange → {0, 8},
 PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 1, 0],
   RGBColor[0, 0, 1]}]
```



From the graph and the table below students can confirm their findings.

```
Table[{x, 2^x, 4^x, 6^x} , {x, -1, 3}] // TableForm
```

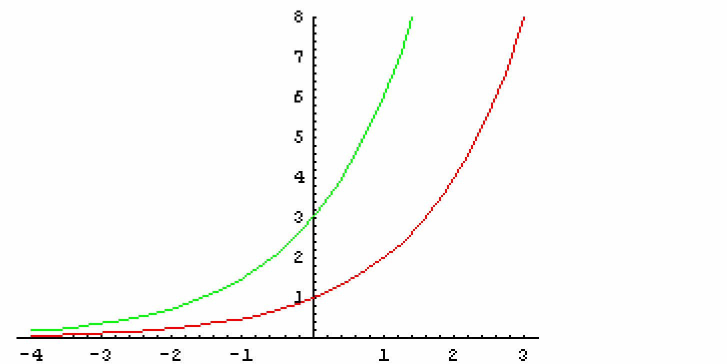
-1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{6}$
0	1	1	1
1	2	4	6
2	4	16	36
3	8	64	216

Structure

They recognise and explain the roles of the relevant constants in the relationships … f(x) = ca^x.

When the exponential function is of the form $y = ca^x$, the coefficient c has the effect of dilating the graph from the x axis, without changing the basic shape of the graph or the asymptote.

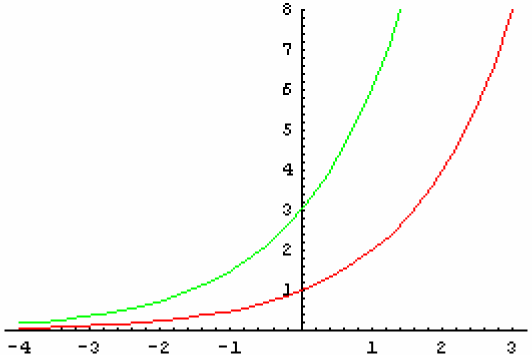
```
Plot[{2^x, 3 × 2^x}, {x, -4, 3}, PlotRange → {0, 8},
 PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 1, 0]}]
```



- the y-intercept (0,1) becomes (0,3) and in general, the y–intercept becomes $(0,c)$;
- the equation of the asymptote is still the same $y = 0$

A useful benefit of CAS is that students can readily plot graphs using conventional function notation directly, which helps to emphasise the effect of varying a parameter on a given transformation.

```
f[x_] := 2^x

Plot[{f[x], 3 f[x]}, {x, -4, 3}, PlotRange → {0, 8},
 PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 1, 0]}]
```



This can be investigated further by considering: $f(x)+k$, $f(x-h)$, $f(x-h)+k$ and $f(c(x-h))+k$.

Working Mathematically

Students choose, use and develop mathematical models and procedures to investigate and solve problems set in a wide range of practical, theoretical and historical context …

- Population change in Australia

It has been said that Australia's population has been increased exponentially since the turn of the century. In fact, some sources claim that Australia's population increases by 1%. Students are offered to establish whether existing data can be said to follow a relationship which is approximately exponential.

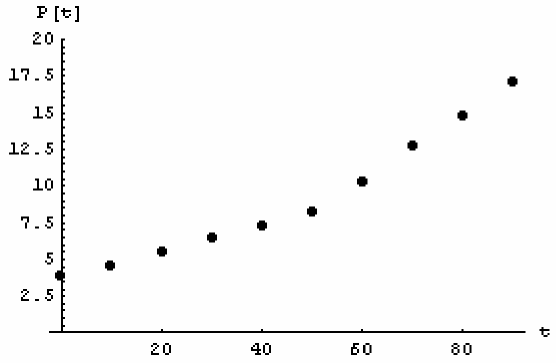
The following table gives the population of Australia (in millions) at the start of each decade from 1900 to 2000.

Year	1990	1910	1920	1930	1940	1950	1960	1970	1980	1990	2000
Number of years (t) elapsed since 1900	0	10	20	30	40	50	60	70	80	90	100
Population P(t) in millions	3.8	4.5	5.4	6.4	7.3	8.2	10.3	12.7	14.7	17.1	

a. Plot the data

```
In[1]:= Data = {{0, 3.8}, {10, 4.5}, {20, 5.4}, {30, 6.4},
          {40, 7.3}, {50, 8.2}, {60, 10.3}, {70, 12.7},
          {80, 14.7}, {90, 17.1}}

Out[1]= {{0, 3.8}, {10, 4.5}, {20, 5.4},
         {30, 6.4}, {40, 7.3}, {50, 8.2}, {60, 10.3},
         {70, 12.7}, {80, 14.7}, {90, 17.1}}
```

```
In[2]:= DataPlot = ListPlot[Data, PlotRange → {0, 20},
          AxesLabel → {t, P[t]}, PlotStyle → PointSize[0.025]]
```




```
Out[2]= - Graphics -
```

The function which we can use to model the population of Australia will have a rule of the form: $P(t) = P(0) \times a^t$

$P(t)$ is dependent variable
$P(0)$ is the initial value for $P(t)$
a is the base and symbolises the extent of the growth

b. From the table read off the 'initial' population of Australia $P(0)$.
The base a is a measure of how quickly the function changes.

c. Using the information above, for Australia's population growth, give an estimate for a.

d. Using your answer for $P(0)$ and a, write the rule of an exponential function that might fit the data well by eye.
$P(t) = P(0) \times a^t$
Or $P(t) =$ ________ × (______) ^{t}

e. Plot the function from **d.** By eye judge how well your rule fits the data.

f. Try different values for a to find an exponential function which fits the data well by eye. Write down the rule for your exponential function. Check your rule using

Mathematica

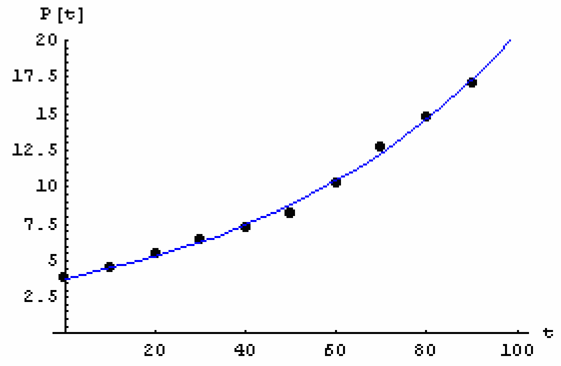
```
In[3]:= FindFit[Data, b a^t, {a, b}, t]

Out[3]= {a → 1.01708, b → 3.75745}
```

The rule of the function is defined according to *a* and *b*

```
In[4]:= P[t_] := 3.8 × 1.017^t
```

Using this function, students can plot the graph:

```
In[5]:= Plot1 = Plot[P[t], {t, 0, 100}, PlotRange → {0, 20},
    PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 0, 1]}];
  Show[DataPlot, Plot1]
```




```
Out[5]= - Graphics -
```

From the graph students can see that the model gives a good approximation of the data. To confirm this result they can evaluate $P(t)$ to determine the Australia's population at a given time, for example $P(0)$:

```
In[6]:= P[0]

Out[6]= 3.8
```

g. Check your rule for 1990 by comparing the population value obtained for the rule and the value listed in the original table.

The same way the population of Australia can be evaluated for different years and compared to the listed value:

```
In[7]:= P[90]

Out[7]= 17.3248
```

The table gives 17.1 million. Students can see that the real data does not fit the model perfectly.

h. Use your fitted function to find the population in 1980 and 1981 (that is $t = 80$ and $t = 81$). Now calculate *the percentage annual change* given by the formula:

$$\text{Percentage annual change} = \frac{P(91) - P(90)}{P(90)} \times 100\%$$

The computations of percentage annual change could be carried out for two successive years using *Mathematica*:

In[8]:= **Percentage annual change** = $\frac{P[91] - P[90]}{P[90]} \times 100$

Out[8]= 1.7

i. Repeat part **h.** for two successive years of your choice:

In[9]:= **Percentage annual change** = $\frac{P[81] - P[80]}{P[80]} \times 100$

Out[9]= 1.7

In[10]:= **Percentage annual change** = $\frac{P[61] - P[60]}{P[60]} \times 100$

Out[10]= 1.7

In general, using function notation we can work out the percentage annual change from any year(t)to the following year (t + 1) as follows.

$$\text{Percentage annual change} = \frac{P(t+1) - P(t)}{P(t)} \times 100\%$$

Using *Mathematica*:

In[11]:= **Percentage annual change** = $\frac{P[t+1] - P[t]}{P[t]} \times 100$

Out[11]= $26.3158\ 1.017^{-t}\ (-3.8\ 1.017^{t} + 3.8\ 1.017^{1+t})$

In[12]:= **Expand**$[26.32 \times 1.017^{-t}\ (-3.8 \times 1.017^{t} + 3.8 \times 1.017^{1+t})]$

Out[12]= 1.70027

j. How do these answers for parts **h.** and **i.** compare?

Students can see that percentage annual change is the same and equal to 1.7, so the students can predict the population for the following year.

This answer gives the same result as the previous calculation.

k. With percentage annual change of 1.7%, and the fact that $P(90) = 17.1$(in millions), predict the population for the years 1991, 1992 and 1993 using *Mathematica.*

To answer this question, students can use the **Solve** commands:

In 1991:

In[13]:= Solve$\left[1.7 == \frac{p - 17.1}{17.1} \times 100, p\right]$

Out[13]= {{p → 17.3907}}

In 1992:

In[14]:= Solve$\left[1.7 == \frac{p - 17.39}{17.39} \times 100, p\right]$

Out[14]= {{p → 17.6856}}

In 1993:

In[15]:= Solve$\left[1.7 == \frac{p - 17.69}{17.69} \times 100, p\right]$

Out[15]= {{p → 17.9907}}

So, in 1991 Australia's population was 17.4 million, in 1992 – 17.7 million and in 1993 – 18.0 million.

l. In 1993 the actual population of Australia was 17.7 million. How far out was your prediction in **k**. Give your answer as a number of persons and as percentage of the actual population.

In[16]:= **18 – 17.7**

Out[16]= 0.3

In[17]:= **0.3 × 1000000**

Out[17]= 300000.

The model predicts 300000 people more then the actual population in1993.

In[18]:= $\frac{0.3}{17.7} \times 100$

Out[18]= 1.69492

The difference between the actual and predicted population of Australia in 1993 were 1.69%.

This project provides students with an insight into the use of mathematics within a real life context. Also they learn about limitation of models and gain skills at interpreting and analysing data.

Pythagoras theorem

At level 6 students are introduced to pythagoras theorem. This is applicable to the Measurement, chance and data, Structure and Number.

Measurement, chance and data

… Students use pythagoras theorem … to obtain lengths of sides … of right-angled triangles.

Structure

Students form and test mathematical conjectures; for example, 'What relationship holds between the lengths of the three sides of a triangle?'

They use irrational numbers … and common surds in calculations in both exact and approximate form.

Number

… They specify decimal rational approximations for square roots of primes, rational numbers that are not perfect squares …

Pythagoras theorem in exact and approximate form.

Example 1

Find the length of the missing side expressed in exact form and approximate correct to two decimal places.

Equation:

$c^2 = a^2 + b^2$

$a = 3$

$b = 6$

$c^2 = 3^2 + 6^2$

$c^2 = 9 + 36$

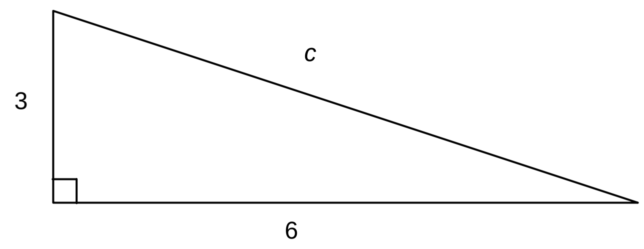
$c^2 = 45$

$c = \sqrt{45}$

$c = 3\sqrt{5}$ – **in exact form**

$c = 6.7082$

$c = 6.71$ – **in approximate form correct to two decimal places**.



Using *Mathematica*:

a. in exact form

```
Solve[c^2 == 3^2 + 6^2, c]

{{c → -3 √5}, {c → 3 √5}}
```

b. in approximate form

```
Solve[c^2 == 3^2 + 6^2, c] // N

{{c → -6.7082}, {c → 6.7082}}
```

Students have to analyse the answers given and choose only the positive answer.

Example 2

A ladder 3.5 metres long is leaning against the vertical wall with the base 1.5 metres from the bottom of the wall on horizontal ground. How high up the wall does the ladder reach? Give your answer correct to two decimal places.

Solution:

Use a pythagoras theorem

$c^2 = a^2 + b^2$

Let $a = 1.5$ m
$b = h$
$c = 3.5$ m

$3.5^2 = 1.5^2 + h^2$

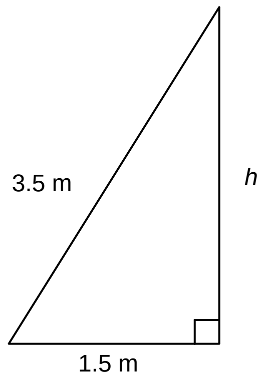
$h^2 = 3.5^2 - 1.5^2$

$h^2 = 12.25 - 2.25$

$h^2 = 10$

$h = \sqrt{10}$

$h = 3.16$



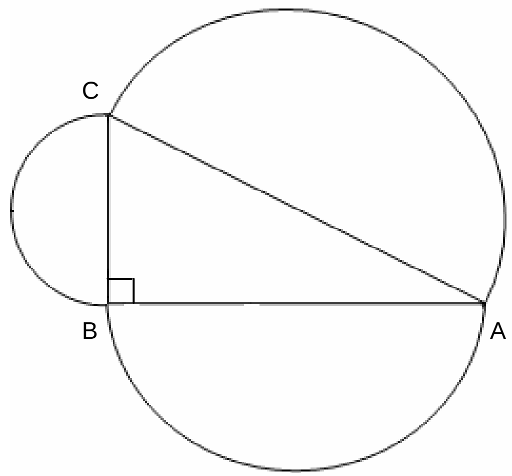
Using *Mathematica:*

```
Solve[3.5^2 == 1.5^2 + h^2, h]

{{h → -3.16228}, {h → 3.16228}}
```

So, $h = 3.16$ m

Exploratory activity

Students were introduced to pythagoras theorem using a geometric interpretation which involved comparing the sum of the areas of squares constructed on the perpendicular side lengths with the area of a square constructed on the hypotenuse. This exploratory activity involves student investigating whether this rule can be applied to other geometric shapes such as semi-circles, equilateral and isosceles triangles and polygons. This activity encompasses skills learnt in all dimensions.



1. Construct a semi-circle on each side of a right angle triangle.

2. Compare the areas of the semi-circles and discuss your results.

3. Replace the semi-circles with the following shapes:
   a. equilateral triangles
   b. isosceles triangles
   c. rectangles
   d. pentagons

   and repeat the activity described in part 1.

4. Why does pythagoras rule work for some shapes and not others?

Note: students can consider symmetry properties and proportionality.

Difference equations

Structure

Students distinguish between linear [and] quadratic … types of functions by testing for constant first difference, constant second difference or constant ratio between consecutive terms (*for example to distinguish between the functions described by the sets of ordered pairs*)
{(1, 2), (2, 4), (3, 6), (4, 8) …}; {(1, 2), (2, 4), (3, 8), (4, 14) …}; and {(1, 2), (2, 4), (3, 8), (4, 16) …}). *They use and interpret the functions in modelling a range of contexts.*

The technique of finite differences provides a very useful method for problem solving in Mathematics. It enables us to find complex rules for numerical patterns. This activity involves finding polynomial models for number patterns.

Part A.

1. Introductory activity – Finding linear models.

Example

Using *Mathematica* create a table for $y = 2x + 3$

```
Table[{x, 2 x + 5}, {x, 0, 6}] // TableForm

0    5
1    7
2    9
3    11
4    13
5    15
6    17
```

Notice that there is a *common difference* of 2 between each term in the table. Using *Mathematica* we can subtract successive terms in the sequence simultaneously to show this *common difference.*

```
{7, 9, 11, 13, 15, 17} - {5, 7, 9, 11, 13, 15} //
 TableForm

2
2
2
2
2
2
```

What does this common difference correspond to in the rule?

a. Compare the constant difference for the pattern with the rule $y = 3x + 5$ and that for the pattern with the rule $y = 2x + 3$.

- Using *Mathematica* create a table of values for the first 6 terms.
- Subtract successive terms in the table. What do you notice?

b. Now try a general case. Find the first 7 terms and the common difference between each term $y = ax + b$

```
Table[{x, a x + b}, {x, 0, 6}] // TableForm

0      b
1      a + b
2      2 a + b
3      3 a + b
4      4 a + b
5      5 a + b
6      6 a + b
```

The *first* difference between successive *y*-values is common and equal to *a* that show the relationship between *x* and *y* is linear.

```
{a + b, 2 a + b, 3 a + b, 4 a + b, 5 a + b, 6 a + b} -
  {b, a + b, 2 a + b, 3 a + b, 4 a + b, 5 a + b} // TableForm

a
a
a
a
a
a
```

x	$y = ax + b$	Differences	
		First differences	Second differences
0	b		
		a	
1	$a + b$		0
		a	
2	$2a + b$		0
		a	
3	$3a + b$		0
		a	
4	$4a + b$		0
		a	
5	$5a + b$		0
		a	
6	$6a + b$		

c. Find the rule for the following number patterns:
   i. 7, 12, 17, 22 …
   ii. 4, -3, -10, -17 …

To find the rule students have to realise that:

$$\begin{cases} a = 12 - 7 \\ a + b = 7 \end{cases}$$

Use a Solve command to find *a* and *b*

```
Solve[{a == 12 - 7, a + b == 7}, {a, b}]

{{a → 5, b → 2}}
```

So $y = 5x + 2$

2. Finding Quadratic Models

a. Using *Mathematica* create a table of values of

$y = x^2$, $\{x, 0, 6\}$

b. Use the table in part **a.** to find news differences until a sequence of constant differences is obtained.

c. Compare the constant difference for the pattern with the rule

$y = 3x^2$

Can you see a connection between the constant difference and the coefficients of particular terms in each of the equations?

d. For the pattern with rule $y = ax^2$

e. Repeat this process for

- $y = 3x^2 + 7$
- $y = 3x^2 + 2x$
- $y = 5x^2 + 3x + 6$

f. Try a general case $y = ax^2 + bx + c$.

We begin the difference table by evaluating the quadratic equation for *x* values of 0, 1, 2, 3, 4, 5 and 6.

```
Table[{x, a x^2 + b x + c}, {x, 0, 6}] // TableForm

0    c
1    a + b + c
2    4 a + 2 b + c
3    9 a + 3 b + c
4    16 a + 4 b + c
5    25 a + 5 b + c
6    36 a + 6 b + c
```

The next step is to find the difference between successive *y*-values.

```
{4 a + 2 b + c, 9 a + 3 b + c, 16 a + 4 b + c, 25 a + 5 b + c,
   36 a + 6 b + c} - {a + b + c, 4 a + 2 b + c, 9 a + 3 b + c,
   16 a + 4 b + c, 25 a + 5 b + c} // TableForm

3 a + b
5 a + b
7 a + b
9 a + b
11 a + b
```

The difference between successive *y*-values is not constant therefore we need to find the second differences – the difference between successive first differences.

```
{5 a + b, 7 a + b, 9 a + b, 11 a + b} -
  {3 a + b, 5 a + b, 7 a + b, 9 a + b} // TableForm

2 a
2 a
2 a
2 a
```

The second differences are constant, so the difference table can be completed.

x	$y = ax^2 + bx + c$	Differences	
		First differences	Second differences
0	c		
		$a+b$	
1	$a+b+c$		$2a$
		$3a+b$	
2	$4a+2b+c$		$2a$
		$5a+b$	
3	$9a+3b+c$		$2a$
		$7a+b$	
4	$16a+4b+c$		$2a$
		$9a+b$	
5	$25a+5b+c$		$2a$
		$11a+b$	
6	$36a+6b+c$		

g. Use a difference table method to find the rule for the following:

x	1	2	3	4	5	…
y	6	17	34	57	86	…

To find the rule, students have to establish the relationships from example in Part **f**:

a. find the first difference

```
{17, 34, 57, 86} - {6, 17, 34, 57} // TableForm
11
17
23
29
```

b. find the second difference

```
{17, 23, 29} - {11, 17, 23} // TableForm
6
6
6
```

Students can see that the equation is quadratic because the second difference is constant.

x	$y = ax^2 + bx + c$	Differences	
		First differences	Second differences
0	1		
		5	
1	6		6
		11	
2	17		6
		17	
3	34		6
		23	
4	57		6
		29	
5	86		

c. Solve the simultaneous equations:

$$\begin{cases} a+b+c=6 \\ 3a+b=17-6 \\ 2a=6 \end{cases}$$

```
Solve[{a + b + c == 6, 3 a + b == 17 - 6, 2 a == 6}, {a, b, c}]

{{a → 3, b → 2, c → 1}}
```

Therefore, the rule for the given pattern is:

$y = 3x^2 + 2x + 1$

3. Finding Cubic Models

Find a cubic equation for the following patterns

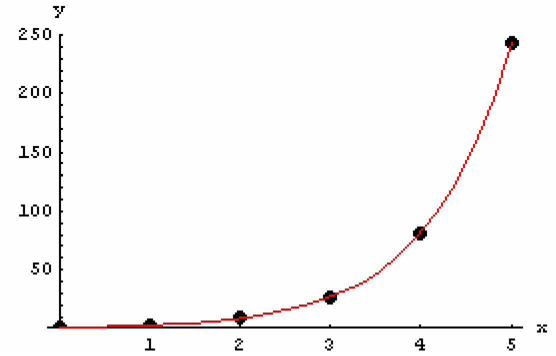
i. -3, 4, 19, 48, 97, 172, 279 ...
ii. 0, 4, 14, 36, 76, 140, 234 ...

4. Finding Exponential Models

Consider the relationship described in the following table:

x	0	1	2	3	4	5
y	1	3	9	27	81	243

The graph of this relationship can be incorrectly described as a part of a parabola.



Setting up a finite difference table, students will be able to establish that the second finite differences are not constant; therefore the relationship is not quadratic. If students continue investigating finite differences they will not be able to find a constant difference.

x	y	Differences				
		First differences	Second differences	Third differences	Forth differences	Fifth differences
0	1					
		2				
1	3		4			
		6		8		
2	9		12		16	
		18		24		32
3	27		36		48	
		54		72		
4	81		108			
		162				
5	243					

Students may recognise a pattern: each y value and each set of terms in each finite difference column is 3 times the previous one: they can recognise *constant ratio between consecutive terms that* is equal to 3.

This indicates that we are multiplying the y value by 3 when x increases by 1.

$x = 0$ $\quad y = 3^0 = 1$
$x = 1$ $\quad y = 3^1 = 3$
$x = 2$ $\quad y = 3^2 = 9$
$x = 3$ $\quad y = 3^3 = 27$
… …

So, it is a new exponential relationship which we can represent by equation $y = 3^x$.

Part B

Problem Solving

I am not sure that it is of great value in life to know how many diagonals a n-sided figure has. It is the method rather than the result is valuable
W. W. Sawyer

Problems from *Finite Differences: A Pattern Discovery Approach to Problem Solving,*
Seymore and Shedd, Creative Publications, 1973.

Trigonometry and circular (trigonometric) functions

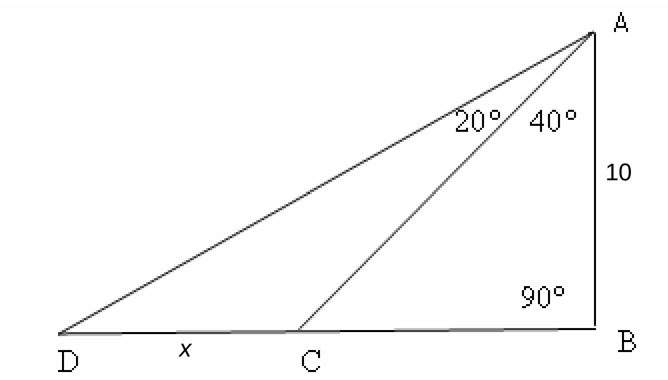
Measurement, chance and data

Students use pythagoras theorem and trigonometric ratios (sine, cosine and tangent) to obtain lengths of sides, angles and the area of right-angled triangles.

They use degrees and radians as units of measurement for angles and convert between units of measurement as appropriate.

Once students have mastered the basic application of trigonometric ratios they can extend their knowledge to more complex problems and solve them using *Mathematica.*

Example 1
Find the value of a pronumeral.



To find x: $x = DB - CB$

To find BC, use triangle ABC.

From the equation $\tan\theta = \frac{O}{H} \Rightarrow \tan 40° = \frac{BC}{10} \Rightarrow BC = 10 \times \tan 40°$

Using *Mathematica*

```
10 Tan[40 °] // N

8.391
```

$BC = 8.39$

To find BD, use triangle ABD

From the equation $\tan\theta = \frac{O}{H} \Rightarrow \tan(20° + 40°) = \frac{BD}{10} \Rightarrow BD = 10 \times \tan 60°$

```
10 Tan[60 °] // N

17.3205
```

$BD = 17.32$

Because $x = DB - CB$

$\Rightarrow x = 17.32 - 8.39$

$x = 8.93$

Or

```
Solve[{Tan[40 °] == a/10, Tan[60 °] == (a + x)/10}, x] // N

{{x → 8.92951}}
```

Introduction to the unit circle and circular (trigonometric) functions

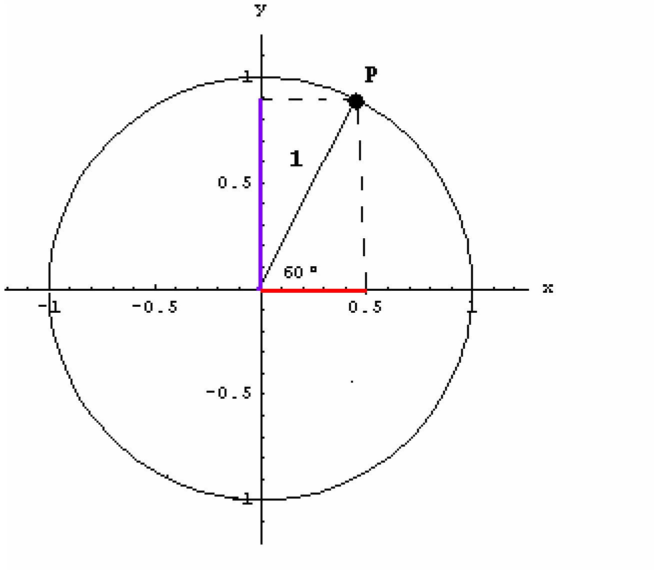
At this stage students can be introduced to the unit circle and the radian measure.

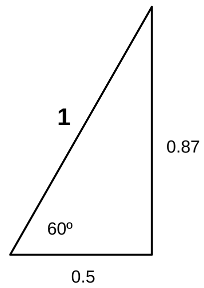
Students are first introduced to the properties of the unit circle (a circle with circumference 2π) and learn how to convert from radians to degrees.

The circle is then transferred into the cartesian plane. Using a graphics package students recognise that the position of the radius can be described as the coordinates of the point of intersection of the radius and circumference of the circle (*x, y*). They can form a right angle triangle with a hypotenuse of 1 (radius) and sides length of *x* and *y*. Students then learn the correspondence between the *x*-axis and the cosine of the angle and the *y*-axis and the sine of the angle.

Example 2

a. Use the unit circle to find the x and y coordinates of point P and hence, cos 60° and sin 60°.





Extracting the right angle triangle from the unit circle students can see that

$$\sin 60° = \frac{0.87}{1} \text{ and}$$

$$\cos 60° = \frac{0.5}{1}$$

Because the hypotenuse is equal to one students can make a connection between the x coordinate and $\cos\theta$ and the y coordinate and $\sin\theta$.

Using the properties of symmetry students can recognise why the sign of the given trigonometric ratios can be positive or negative. It is important for them to know this because when solving trigonometric equations with CAS the answer is given as both positive and negative and they need to be able to interpret the result.

Students also need to be introduced to radian measure because when using *Mathematica* the answers are given in radian measure.

Example 3

a. Convert $50°$ to exact radians, expressing your answer as a multiple of π.

Solution

$$50 \cdot \frac{\pi}{180}$$

$$\frac{5\pi}{18}$$

Answer: $50° = \frac{5\pi}{18}$

b. Convert $\frac{2\pi^c}{5}$ to degree

$$\frac{2\pi}{5} \cdot \frac{180}{\pi}$$

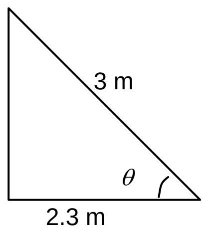
$$72$$

Answer: $72°$

Trigonometric ratios can be used to solve application problems using CAS.

Example 4

The highest point that Jason can get a swing to rise he is 2.3 metres from the swing supports. If the chain of the swing is 3 metres long what angle does the chain make with the horizontal when Jason is at the top of his swing?



To find θ the trigonometric ratio is used $\cos\theta = \frac{Adjacent}{Hypotenuse}$

```
Solve[Cos[θ] == 2.3/3, θ]

{{θ → -0.697163}, {θ → 0.697163}}
```

The equation gives two solutions in radians, so students have to apply their knowledge to convert radians to degrees. Because students are familiar with the unit circle they can choose only the positive solution $\theta = 0.697^c$.

To convert the angle measured in radians $\theta = 0.697^c$ to degree:

```
0.697 180/π

39.9352
```

So $\theta = 39.945°$

The introductory work on the unit circle can be extended to consider graphs of circular (trigonometric) function. At this stage students would plot the basic graphs using points from the unit circle. These graphs can be revisited at the end of the Year 10 when students work with families of curves.

Exploring circular function (trigonometric) graphs

From the unit circle students know that circular (trigonometric) functions are periodic. The sine and cosine of an angle is read off on the *y*- and *x*- axis of the unit circle respectively. At the beginning it will be valuable to ask students to plot graphs by hand and establish the features of $y = \sin x$ and $y = \cos x$: amplitude and period.

It can be done by completing a table of sine or cosine values by using *Mathematica* or reading values of these functions directly from the unit circle.

```
Table[{x, Sin[x]}, {x, 0, 360°, 30°}] // TableForm
```

0	0
30°	$\frac{1}{2}$
60°	$\frac{\sqrt{3}}{2}$
90°	1
120°	$\frac{\sqrt{3}}{2}$
150°	$\frac{1}{2}$
180°	0
210°	$-\frac{1}{2}$
240°	$-\frac{\sqrt{3}}{2}$
270°	-1
300°	$-\frac{\sqrt{3}}{2}$
330°	$-\frac{1}{2}$
360°	0

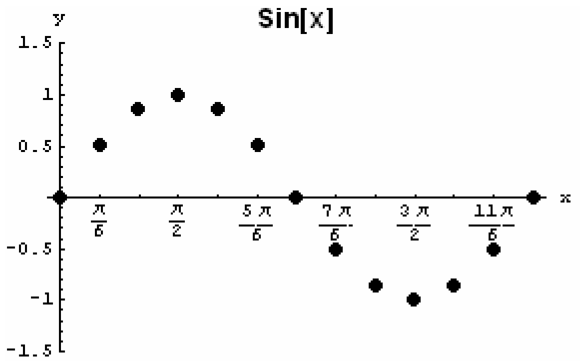
Using radians: (answers can be given in exact and approximate form)

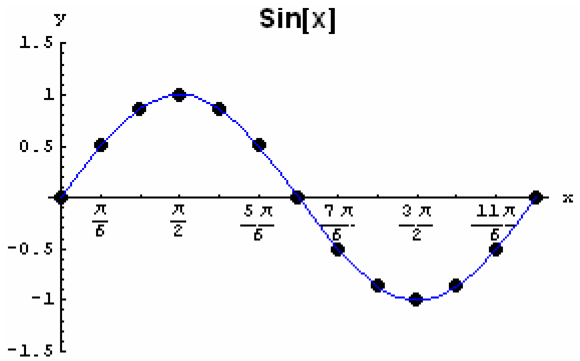
$$\texttt{Table}\left[\{x, \texttt{Sin}[x] \;//\; \texttt{N}\}, \left\{x, 0, 2\pi, \frac{\pi}{6}\right\}\right] \;//\; \texttt{TableForm}$$

0	0.
$\frac{\pi}{6}$	0.5
$\frac{\pi}{3}$	0.866025
$\frac{\pi}{2}$	1.
$\frac{2\pi}{3}$	0.866025
$\frac{5\pi}{6}$	0.5
π	0.
$\frac{7\pi}{6}$	-0.5
$\frac{4\pi}{3}$	-0.866025
$\frac{3\pi}{2}$	-1.
$\frac{5\pi}{3}$	-0.866025
$\frac{11\pi}{6}$	-0.5
2π	0.

0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$
π	0
$\frac{7\pi}{6}$	$-\frac{1}{2}$
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{3\pi}{2}$	-1
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{11\pi}{6}$	$-\frac{1}{2}$
2π	0

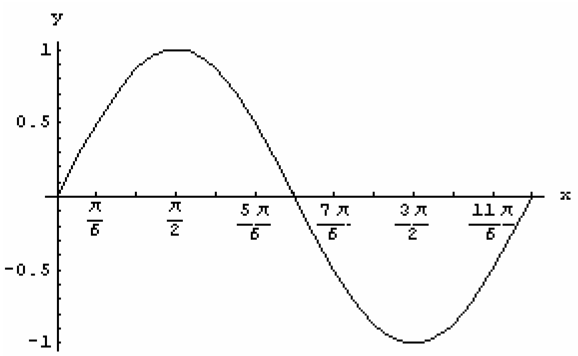
Using the information from the table ask students to plot the graph $y = \sin x$ for these values, and then extend to the continuous function on the graph paper.





After this exercise student can use *Mathematica* to sketch the graph.

```
Plot[Sin[x], {x, 0, 2π}]
```



Once students have established the features of circular (trigonometric) functions you may introduce them to different basic transformations.

Example

Sketch the graphs of the following, using *Mathematica*, and describe the difference between each graph and the graph $y = \sin x$.

a. $y = \sin x + 1$

b. $y = \sin x - 3$

c. $y = -\sin x$

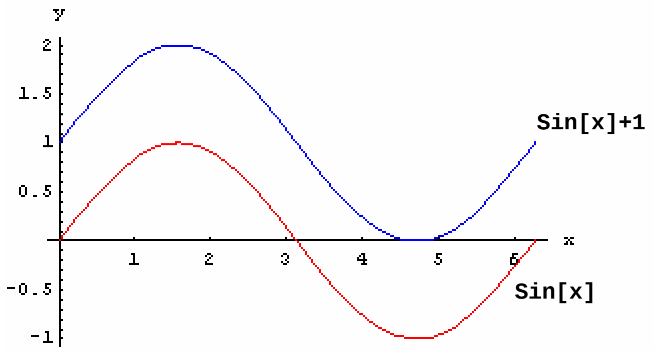
d. $y = \sin 2x$

e. $y = \sin 4x$

Solution

a.

```
Plot[{Sin[x], Sin[x] + 1}, {x, 0, 2 π},
 PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 0, 1]},
 AxesLabel → {x, y}]
```

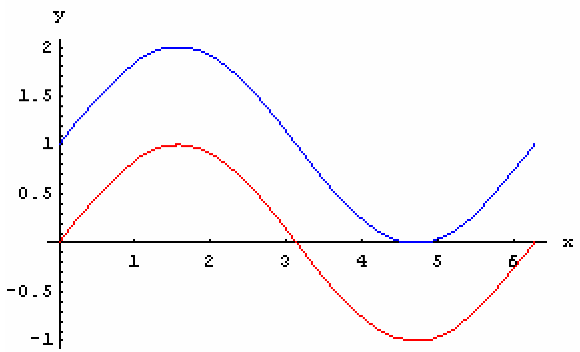


↑ ***One unit vertically*** **up**

From his graph students can see the translation of $y = \sin x$ one unit vertically up. This kind of exercises leads students towards understanding function notation that can be presented:

$f(x) = \sin x \qquad \rightarrow \qquad f(x) + 1$

```
f[x_] := Sin[x]

Plot[{f[x], f[x] + 1}, {x, 0, 2 π},
 PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 0, 1]},
 AxesLabel → {x, y}]
```



Students are familiar with the basic transformations from their work with quadratic graphs and the use of function notation. This gives them the opportunity to see that exactly the same rules apply to all functions. The function notation that the CAS uses is of significant benefit to students. They learn quickly how to use it to perform different transformations.

Through well-chosen exercises students can build up their understanding of the different transformation.

A computer based CAS such as *Mathematica* is not only an excellent tool to perform transformations but its graphing capabilities also allows students to see clearly the connection between the original function and it's image.

We have found that CAS is a valuable tool for the teaching and learning mathematics at the junior secondary level. In particular, it can enhance students understanding of algebraic expressions, allowing them to explore properties such as similar expressions, simplification of algebraic fractions and the expansion and factorisation of polynomials. CAS is also useful in helping students to recognise the connection between the algebraic and graphical methods.

CAS technology allows students to extend their awareness of difference type of graphs and solve more complicated and general equations that they would not normally encounter without use of such technology.