Essential Learning
Prep to Year 10
Mathematics Curriculum Area

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VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victoria The Place To Be

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Contents

Section 1: Introduction 4

Rationale 4

Section 2: Key elements of the school mathematics curriculum 6

The aims of school mathematics 6

The central ideas derived from the discipline 13

The domains of study in school mathematics and their key concepts 17

Section 3: Future directions 23

A model of the infrastructure for the school mathematics curriculum P–10 23

What might happen next? 23

Final comments 24

Section 4: References 25

Section 1: Introduction

This paper contends that the school mathematics curriculum should be founded in contemporary views about the discipline of mathematics (including its history and philosophy) and the history and philosophy of mathematics education. These views are based on the submissions of local, national and international mathematicians and mathematics educators.

There is a well established tradition, beginning with Plato and Aristotle, of mathematicians contributing ideas to the design and implementation of the school mathematics curriculum. Since the early years of the twentieth century, the influence of mathematicians on the school curriculum has become more pervasive, as expressed, for example, in the contributions of Felix Klein and Alfred North Whitehead. It is acknowledged that while the views expressed here draw on a historical heritage of several millennia across various cultures, they are formed within the socio-political context of the time.

Rationale

While school mathematics must necessarily be a carefully selected subset of the discipline it must nonetheless embody the spirit of the discipline and its ubiquitous role in everyone's lives. The people involved in the discipline are continually building on its deep cultural heritage and developing new areas of mathematics. These developments have burgeoned in the last two hundred years. So the mathematics selected for inclusion in the school mathematics curriculum today must initiate children into the wealth of the culture of mathematics that is part of their wider cultural heritage. As such the school mathematics curriculum is an introduction to the ways people develop and use mathematics today to make sense of their lives and the world they live in.

Part of the spirit of mathematics that must be embodied in the curriculum is that it is primarily a human endeavor. The nature of this endeavor can be the stuff of drama and literature. The narrative of this endeavor could refer to imaginative and creative people who had faith in what was intrinsically true for them, who faced moral and ethical dilemmas, met dead-ends and criticism and felt despair, yet often persevered to find that their ideas, while not immediately applicable, evolved into other ideas or contributed to other areas of mathematics, other disciplines, technologies or socio-political contexts. The spirit of these endeavors is essentially about people contributing to their own lives and to the society in which they live. The school mathematics curriculum can convey this spirit in its discourse and practice especially in the primary and middle schools where the pressure of hurdle examinations is not so strong.

As with most human endeavors what was considered to be sophisticated in mathematics at one time (and so available to only some people) is later taken to be part of everyday activity for all members of a society. This is the case with arithmetic in ancient times and computation today.

> *It is a regular outcome of the activity of mathematicians, throughout the centuries, that formerly difficult questions become easy, easier [or different] ones, and that mathematical tools which were at first the privilege of experts sooner or later become available to novices* (Chevallard, 1989)

This means that much of the mathematics that people use in everyday life is embodied in the tools they use and is, as such, invisible. This is the case today with, for example, calculators and various computer-software packages and will, one might expect, become the case with graphics calculators and computer algebra systems in the near future. This sort of invisibility conceals the power and ‘unreasonable effectiveness’ (Wigner, 1960) of mathematics to describe the world and contribute to people’s lives. On the other hand these technologies enable teachers and students to explore and ‘experiment’ (Stewart, 1996) with mathematics in their classrooms. However, the danger in such invisibility is that mathematics is valued in society primarily for its undoubtable utility. This is the sort of focus that allowed problem solving to become more prominent than problem posing in school curricula worldwide in the 1980s. Mathematics is of more value in a school curriculum and to society than for its so-called utility.

The school mathematics curriculum provides a platform for teachers and learners (and teachers as learners) to develop their mathematical capabilities beyond those required for functional utility in society. This can be achieved by developing their inclinations, sensitivities and abilities to appreciate mathematics. This idea is elaborated in the section entitled ‘The aims of school mathematics’ (See Section 2). There is a worthwhile analogy between mathematical capability and the appreciation of mathematics, on the one hand, and the study of language and literature, on the other.

There are parallels between mathematical capability and being able to use language effectively for oral and written communication. Mathematical appreciation resembles the study of literature, in that it is concerned with the significance of mathematics as an element of culture and history. Just as great texts are seen in their cultural and historical context, so the artifacts of mathematics are similarly understood. Such appreciation involves an awareness of how mathematics permeates every corner of human life and culture.

Is that kind of appreciation a feasible focus for school mathematics? The contributors to this current review of the essentials of mathematics P–10 believe that it is. One contributor put it this way:

> *I believe that it is our conception of mathematics in school that limits learners’ appreciation, not their capacities. One must be careful about prejudging what is appropriate or accessible to schoolchildren.*

For example,

> *One of the arguments against encouraging the appreciation of big ideas like infinity is that it is too difficult for schoolchildren. However many an interested 8 year old will happily discuss the infinite size of space, or the never-ending nature of the natural numbers.*

Section 2: Key elements of the school mathematics curriculum

The school mathematics curriculum is based on three key elements:

- The aims of school mathematics.
- Central ideas derived from the discipline.
- Areas of study in school mathematics, and the key concepts therein.

The aims of school mathematics

The aims of school mathematics should be considered in the contexts of:

- the aims of general education in schools
- the expectations those with the power to set these aims have of the people who are the subjects of these aims
- the need to balance the requirements and abilities of the general population against the desires of professional mathematicians.

It is important to provide for the latter since they will be the ones to lead and teach the next generation. For the general population, the relevance and accessibility of the mathematics taught will be paramount.

Like the aims of general education, the aims of school mathematics education are vitally concerned with the social, cultural and economic aspects of a liberal, civil, just and democratic society and how these contribute to the development and dynamics of such a society. The aims of school mathematics are therefore formed in a reciprocal relationship between those who:

- experience the universality of mathematics (for example, all professions need a mathematical training and background related to their way of thinking)
- are delegated to represent the government of the day (for example, the VCAA)
- work in and teach the discipline.

A similar reciprocity exists in varying degrees between mathematics and other disciplines and subject areas in the curriculum. These relationships need to be developed and made explicit in the school curriculum, for example, in the context of integrated curriculum or cross-curriculum activities. The area of data gathering, representation and manipulation, as well as being recognised as an established part of the mathematics curriculum, is also part of other areas of the curriculum. This area is a cross-curriculum activity–data sets could be revisited, expanded and reinterpreted in different areas of the curriculum as learners grow in curiosity and understanding.

The aims of school mathematics provide principles that guide:

- the selection of mathematical experiences, concepts, processes and contexts for inclusion in school mathematics curriculum
- contingent decisions about a multiplicity of factors including student age, aptitudes, interests, intended future study or work, cultural contexts and needs.

The nature of mathematics as a multi-centred discipline (that is, it is not seen as being fixed, unified and monolithic) strongly supports the idea of a multi-centred, multi-natured mathematics curriculum whose variations are determined by differing student needs.

Table 1 (see page 8) contains a description of the contemporary aims of school mathematics. These aims fulfil the criteria described earlier in this section. They describe the objectives for teaching and learning mathematics in our society and the mathematical capabilities associated with these objectives. These capabilities represent the best intentions of one generation for the next in terms of mathematics education. They are principles that can guide and inform the practices of curriculum construction and of teaching and learning. The aims often work in tandem with each other, for example, aims 5 and 6 support the intentions of aims 2 and 4.

Publications such as the CSF II (VCAA, 2002) and the VCE Mathematics Study Design (Board of Studies, 1999) and information about numeracy (see, for example, CSF II, p. 8, also *Middle Years Numeracy Research Project*) provide specific information for teachers and school communities as they elaborate aims 1, 2 and 3. These aims are not necessarily mutually exclusive and are usually matters of emphasis and depth for teachers and school communities.

Aims 4 to 7 inclusive refer to the contribution that school mathematics can make to human lives, individually and collectively. These are deemed to be important for all children and adults. They are in a sense the creed which guides teachers and (with increasing experience with mathematics) learners as they engage in their day-to-day mathematical activity in classrooms. Aims 4, 5 and especially 7 are important in the day-to-day practice of teachers. The threads that draw these aims together have to with thinking and reasoning in mathematical and metacognitive ways. The ability to think mathematically is one of the best gifts we can give our children. The capabilities associated with aims 4, 6, and especially 7 should be a main focus in the professional development associated with the introduction of the essentials of mathematics P–10 reforms in schools.

Table 1: Different mathematical teaching and learning aims and capabilities

Aims	Associated mathematical capabilities
1. Utilitarian knowledge For some or all? – a basic and minimal requirement for all at the end of schooling.	To be able to demonstrate useful mathematical and numeracy skills adequate for successful general employment and functioning in society.
2. Practical, work-related knowledge For some or all? – not necessary for all persons. A strong case can be made for schools to provide the basic understanding and general equipment upon which further specialist knowledge and skills can be built.	To be able to solve practical problems with mathematics, especially industry and work-centred problems. This capability also works in tandem with Aims 5 and 6 below.
3. Advanced specialist knowledge For some or all? – not a necessary goal for all adults. It is needed by a minority of students for further study in mathematics or mathematics dependent further studies, according to their informed choices.	To have an understanding of and capabilities in advanced mathematics, with specialist knowledge beyond standard school mathematics (including both advanced high school specialist study of mathematics, and knowledge of university and research mathematics, as appropriate).
4. Mathematical problem posing and solving powers. For some or all? – All.	To be able to see mathematical connections and be able to deploy mathematical knowledge and powers in both posing and solving mathematical problems. This capability also works in tandem with Aims 5 and 6. Problem posing is the crux of creativity in mathematics at all levels.

Aims	Associated mathematical capabilities
5. Mathematical confidence. For some or all? – All.	To be confident in one's personal knowledge of mathematics, to feel able to deploy it, and to feel able to acquire new knowledge and skills when needed. This aim was endorsed in a number of the invited papers especially in relation to students being life-long learners empowered to learn how to learn mathematics so they will be able to venture into unknown areas and use what they learn. It is important to note that mathematical confidence is associated with attitudes to mathematics. However, positive attitudes cannot be taught directly. They are an indirect product of many factors, including personal, familial, social and cultural factors. One of the most important contributors to attitudes is the means and style by which mathematics is taught and learned, how much learner interest is seized in the classroom, and how emotionally safe the learner perceives the classroom is as a learning environment. It is better for those not involved in Aim 3 to have mathematical confidence and appreciation (Aim 6) rather than specific advanced mathematical knowledge.
6. Social empowerment through mathematics (Critical citizenship) For some or all? – All.	To be empowered through knowledge of mathematics as a highly numerate critical citizen in society, able to use this knowledge in social and political realms of activity, both near (local) and far (global). Achievement of Aim 4 is of central import in the achievement of this objective because it involves taking control of, and setting the agenda oneself, rather than solely reacting to the agendas set by others

Aims	Associated mathematical capabilities
7. Appreciation of mathematics. For some or all? – All.	7.0 To have an appreciation of mathematics as a discipline including its structure, subspecialisms, the history of mathematics and the role of mathematics in culture and society in general. 7.1 Appreciating the role of mathematics in life and work This involves being aware of how, and the extent to which, mathematical thinking permeates everyday and shopfloor life and current affairs. This is not just utilitarian knowledge of mathematics but also a conception of how important mathematics is in commerce, economics (e.g. the stock market), telecommunications, information and communication technology, the fields of health and biology (especially molecular biology) and weather fore-casting, and so on. It involves awareness of the role mathematics plays in representing, coding and displaying information as it is used in society. As mentioned earlier it also involves an awareness of how complex mathematics is forever becoming more central to, but more deeply and invisibly embedded in, all aspects of our daily life and experience. Without a sense of this we cannot maintain proper control over our life and day-to-day affairs, let alone appreciate the role of mathematics in our employment. 7.2 Appreciating the role of mathematics in culture Having a sense of mathematics as a central element of culture, art and life, present and past, which permeates and underpins science, technology and all aspects of human culture is a central part of mathematical appreciation.

Aims	Associated mathematical capabilities
	7.3 Appreciating the role of mathematics in critical citizenship. This appreciation relates to Objective 6. It includes critically understanding the uses of mathematics in society: to identify, interpret, evaluate and critique the mathematics embedded in social, commercial and political systems and claims, from advertisements, for example, in the financial sector, to government and interest-group pronouncements. Mathematics is very widely used to express and support claims and its use lends an authority to advertisements, reports and pronouncements. Every citizen in modern society needs to be able to analyse, question, critique and understand the limits of validity of such uses, and where necessary reject spurious or misleading claims. This need is both to protect the rights and interests of citizens themselves, and also to protect the more vulnerable in society, unable to do this for themselves. Ultimately, such a capability is a vital bulwark in protecting democracy and the values of a humanistic and civilised society. 7.4 Appreciating mathematics as a discipline Since virtually everybody in the developed world spends many years studying mathematics by the end of their studies they should have some conception, even if limited, of mathematics as a discipline. Mathematics is a central element of culture and more people need to realise that there is much more to mathematics than number and what is taught in school. 7.5 Appreciating the role of history in mathematics (and mathematics in history) Being aware of the historical development of mathematics, the social contexts of the origins of mathematical concepts, its symbolism, theories and problems is important in itself, as well as an enhancement of the study of mathematics.

Aims	Associated mathematical capabilities
	7.6 Appreciating philosophical dimensions of mathematics Understanding the ways that mathematical knowledge is established and validated through proof, but also understanding that there are multiple views of the nature of mathematics and controversy over the philosophical foundations of its knowledge is another part of mathematical appreciation. 7.7 Appreciating big ideas in mathematics Mathematical appreciation requires qualitatively understanding some of the big ideas of mathematics in their purest forms. (See page 13 ‘Central ideas derived from the discipline’)

Consideration of the aims of school mathematics provides the guiding principles for exploring the other two essential elements of a school mathematics curriculum: the central ideas derived from the discipline, and the areas of study in school mathematics.

The central ideas are at the heart of each and all of the domains and work in concert with the key concepts in each domain.

The connections between the central ideas and the key domain concepts provide the impetus and depth of knowledge required for designing, implementing and evaluating learning experiences for pupils and assessing student progress.

The central ideas derived from the discipline

These ideas are described below. They are considered by those people that work in and with the discipline to be valid and appropriate for a school mathematics curriculum. They are the 'big ideas' referred to in aim 7 in Table 1.

Computation, broadly interpreted, can be regarded as the process of carrying out particular calculations, manipulations and constructions such as evaluating the value of a real function at a specific value in its domain, differentiating an algebraic expression, or constructing an equilateral triangle. First and foremost with computation, is the understanding of what is being computed, not the process of computation itself.

Proof, broadly interpreted, can be regarded as the process of **principled reasoning** in general, for example, establishing Pythagoras' theorem, deducing the quadratic formula, or the fundamental theorem of calculus. As soon as we ask 'why?' we start proving (Buchberger, 1999).

Proof is an essential part of mathematical activity because it involves convincing and justifying to people that something is true. Proof in school mathematics can be construed as a 'way of thinking' directed toward explaining, convincing and understanding (VCAA, Reasoning and Strategies, 2002).

Today mathematicians, while continuing to affirm the key role of proof, are becoming much more explicit about their use of experiments and the reasoning therein and are publishing the results of their experiments even without rigorous proofs even though they agree experiments are not proofs (Stewart, 1996). Thus mathematics is always, as it were, a work in progress.

Most of the contributors to this review of the essentials of mathematics P–10 emphasise the central place of **'mathematical reasoning'** and **'mathematical thinking'** in the school mathematics curriculum. The heart of mathematics is not mathematical information but the reasoning. Information should not be the fundamental goal of teaching in mathematics. Indeed many of the contributors to this review support the view that the vehicle (specific topic) is less important than the mathematical thinking it encourages.

Mathematical thinking involves:

- making conjectures
- testing the truth of conjectures, for example, by posing counter-examples
- convincing the thinker and others as to whether a conjecture is plausible, justifiable or valid, which leads to the notion of developing proofs for conjectures for which there do not

appear to be any counter examples. The challenge here is to see the difference between several special cases that do fit and an argument showing that all cases must fit a conjecture

- specialising (this involves, for example, finding examples, special cases, counter-examples and emerging patterns) and generalising (this is about searching for and describing regularity and common aspects in a range of settings associated with the context-at-hand)
- abstracting (takes the regularities formed as generalisations and forms them into common structures and broad classes) and instantiating (involves finding instances that can be used to check an abstraction)
- reifying, or making things into objects for further investigation.

Computation and proof should be developed together. These central ideas bring together two main traditions in mathematics, that is, of algorithm and computation, and proof and deduction.

Algorithms are the means of computation and are now today's lifeblood as functions were in the nineteenth and early twentieth centuries. Together with each algorithm one should have a proof that the algorithm does indeed do what it is supposed to do. Computer users are aware that algorithms, such as the ones in the software that we use everyday, do not always produce the answer or behavior that they should. Proofs give understanding, or at least they should; computations give results.

Proof and computation are also related in that generalised computation often leads to proof and, conversely, proof can also be seen as a particular type of computation.

The next group of central ideas are also associated with computation and proof.

First, consider the idea of **recursion**. Recursion refers to processes that are never ending and which often form part of the structure of an algorithm. The roots of this idea can be found, for example, in the riddle 'If you have two wishes what is the second? Two more wishes' (Papert 1980). Recursion is a powerful tool in contemporary mathematics, for example, in terms of modeling the idea of **chaos** and in the field of **metamathematics**. In metamathematics proof itself is considered to be the subject of investigation and consideration of ideal computing devices used (specifically the size of a computer program in terms of information-size rather than run-time) to develop the idea of 'incompleteness'. This idea says that there will always be things that escape the power of formal axiomatic mathematical reasoning, and things that escape the power of any current type of computer program. Likewise the idea of 'undecideability' means that there is no process for deciding whether a given formula is true or not in advance (Chaitin, 2000).

Next in this group of central ideas associated with computation and proof are the ideas of mathematical truth, certainty (and hence uncertainty) and paradox. A mathematical truth can take the form of an axiom (a 'given' or a 'self-evident truth') or the outcome of a process of proof (a theorem that may subsequently be used as an axiom in another context). In both cases the essence of mathematical truth is about people understanding where their reasoning has come from and being confident that the results can be applied elsewhere. Thus mathematical truth is associated with the notions of genesis and certainty in mathematics. How does mathematical knowledge come to be, and on what basis is confidence in the veracity of mathematical knowledge warranted? Is the Appel and Haken computer assisted proof of the 4-colour theorem in graph theory 'acceptable', epistemologically and/or aesthetically? (Appel and Haken 1977). What would/should be the significance afforded to proofs developed or discovered or constructed by a mathematically able artificial intelligence or ‘automated reasoning’ software?

The central ideas of **randomness** and **inference** are also associated with the ideas of **certainty** and **uncertainty**. Randomness refers to processes that are irregular, non-repetitive, or haphazard. The idea of randomness leads to the ideas of **chance,** which is quantified in terms of **probability**. **Inference** deals with drawing conclusions and making predictions based on data and chance. Few things in life are certain; therefore most of the situations involving these central ideas deal with **uncertainty**. So the conclusions based on inferences are classified as **stable** if they are based on **long-term outcomes** and **unpredictable** if they are based on **short-term outcomes**.

The idea of **paradox** is behind many of the major advances in mathematics. Examples include the discovery of irrational numbers by Hippasus, the problems of infinitesimals in the nineteenth century and Cantor’s paradoxes in set theory. From these came, eventually, Cartesian geometry, the modern foundations of analysis and the continuing vitality of set theory and mathematical logic. In these last two cases the paradoxes are still not satisfactorily settled for everyone.

Another central idea implied by notions of ‘never ending’ or ‘going on forever’ in recursion is the idea of **infinity**. There are three (at least) aspects of infinity to consider. First, the possibility of proceeding indefinitely, as in counting the natural numbers and in apprehending non-terminating decimals. This notion of ‘infinite ascent’ opens up possibilities for exploration. Second, the (potentially) infinite repetition of the same calculation produces the same result. This is what gives people confidence in mathematics. It also points to another central idea, that of **invariance,** other examples of which are symmetry and congruence. The third aspect of infinity is that if infinite ascent is possible for the natural numbers, then infinite descent is not. This simple, intuitive observation is the heart of many arguments, such as irrationality proofs and special cases of Fermat’s Last Theorem. If the impossibility of infinite descent is taught as a separate principle the unnerving aspect of *reduction ad absurdum* proofs is isolated away from the mathematical calculations.

The idea of **composition** is pervasive in mathematics from the early learning of number though to the composition of functions in later years. For example, in the earlier years ‘10 of these is 1 of those’, and ultimately, that (10 of (10 of these is 1 of those)) … and … $^{1}/_{10}$ of ($^{1}/_{10}$ of ($^{1}/_{10}$ of these is 1 of those)).

Mathematical structure is an overarching, unifying idea whose scope ranges from patterns, maps, navigating the web and understanding an argument to computer programs and groups, rings and fields in algebra. In terms of computer programs the recursive processes therein are structures within a structure. Thus the idea of mathematical structure straddles the arenas of **continuous** and **discrete mathematics**. Continuous mathematics dominates the middle and upper levels of mathematics curriculum because it underpins the study of calculus. However pressure may come for the inclusion of more discrete mathematics, say in the middle levels of the curriculum, because of the growth in the use of discrete mathematics in, for example, the areas of biology and health (See Aim 7.1). Indeed the arena of discrete structures has expanded rapidly in the last half of the twentieth century for instance in the context of, for example, graphs, networks and theories of languages.

The central ideas of **equal** (equality) and **equivalence** are staples in the discourse of mathematics from the early years of mathematics education. They are quite sophisticated ideas. They apply in many areas of mathematics and need to be gradually and sensitively developed in the school mathematics curriculum. **Equal**, in general terms, refers to the ideas of alikeness or sameness between properties or quantities. It sits between the notions of 'more than' and 'less than'. For example, in geometry, equal can be used to denote exact agreement with respect to some particular property, but not necessarily congruence. Thus triangles with equal altitudes and equal bases can be referred to as being equal in the sense that the area in each case is the same. But the triangles may not be equal according to the property of congruence. Equal is also used in describing a relation between two quantities that are alike in any or all senses, the sense in which they are alike being specified. For example, if two functions of a variable are equal for only certain values of the variable then the relation is an **equation**. If the functions are equal for all values of the variable the relation is an **identity**. The term **equivalence** is used in the context of propositions and various sets (for example, sets of numbers or sets of relations). Propositions are said to be equivalent if they are connected by *'if, and only if'*. An equivalence relation is *true* if both propositions are true, or both are false. For example, for equilateral triangles any particular triangle is *either both* equilateral and equiangular or it is *neither* equilateral *nor* equiangular. In terms of relations, ordinary equality and congruence relations such as those referred to above are examples of equivalence relations. The words highlighted in italics above draw attention to the use of **language in mathematics and mathematics in language**.

(Mathematics) is an extension of language and of the mental processes we develop in the course of learning our language, but is also a language that endeavors to stand outside our own particular language by building up its own concepts, structures, vocabulary, grammar, and patterns. This has ramifications for the communication of mathematics.

The next central idea is about **mathematical models and modeling**.

Mathematical models are representations that can be manipulated to assist thinking. These representations include physical objects, concepts (for example, multiplicative models) and arguments and proofs (theorems). When a child in response to being asked to add 7 and 5 reaches for some counters or sticks and proceeds to lay out first seven, then a further five, and then counts them all up, the objects form a model of, and a model for the counting process. As noted earlier, the number line provides a model for depicting the order properties of the numbers and for interpreting addition, subtraction, multiplication and division of signed numbers. Models are like metaphors: exploring the details of a metaphor is like exploring the details of a model, that is, examining its assumptions, checking the accuracy and appropriateness of fit, and its ability to predict.

Mathematical modeling is a process that can be used by students of all ages in practical situations to reveal that which is concealed or what might be (in terms of predictions about situations in the world and the world of mathematics). It requires students to interpret and evaluate their meanings of a range of mathematical processes and tools. However, appreciation of modeling as a way of thinking takes time to develop as does a framework for the process of mathematical modeling. This sort of framework comes out of the experience of doing mathematics and may be used as a tool which guides and informs, rather than as a proforma to be completed or a recipe to be followed.

Finally, many of the contributors to this project to describe the essential of mathematics P–10 emphasised strongly the place of the **history of mathematics** and the idea of **beauty** in mathematics as central ideas in the school mathematics curriculum. Reference to the history of mathematics in the school curriculum not only provides insights into mathematics as a human endeavor but also provides ideas about the design of the school mathematics curriculum and its implementation in the classroom especially in terms of teaching practices.

The presentation of mathematics as a human endeavor must consist of *much* more than a dollop of biography here and there, and a cartoon version of the mathematical explorers' motivations. It must be integral to its teaching. Presenting mathematics in its social context can be extremely helpful, or critical, to explaining the mathematical ideas themselves. Seeing *why* the mathematical ideas were originally considered can be a huge step towards understanding what the mathematical ideas actually are.

Beauty, as the adage says, may be 'in the eye of the beholder' and so more easily experienced than described. However many people experience it in relation to mathematics and the challenge is to communicate this experience to others. Beauty is often described in relation to truth and power. The philosopher John Armstrong (2004) provides the following description of the experience of beauty:

> *[It] consists in finding a spiritual value (truth, happiness, moral ideals) at home in a material setting (rhythm, line, shape, structure, [formula, graph]) and in such a way that, while we contemplate the object, the two seem inseparable. To be human is to experience life under two guises: physical and spiritual –- this how it seems, whatever the underlying facts. Thus experience of beauty is a reflection, as it were, of being human.*

In closing this section the point can be made that each of these central ideas, together with aims presented in Table 1 (see page 8), may be used as criteria for answering questions from students about the purpose and nature of mathematics in the school curriculum and their lives in general. In fact, teachers need to frame questions related to these criteria (for example, 'Is there a mathematical way of thinking?') and use them periodically so that the students can develop overviews and 'bigger pictures' in their understandings of mathematics.

The domains of study in school mathematics and their key concepts

The five domains represent both traditional and contemporary areas of study in the discipline. The traditional areas are Number, Space and Measurement (including Data and Chance) with the more contemporary ones having the titles 'Structure' and 'Working Mathematically'. 'Structure' is directly related to the central idea of Mathematical Structure and like the 'Working Mathematically' domain relates to the other domains while being areas of study in their own right. The 'Working Mathematically' domain expands the information presented under this heading on page 6 in the CSF II. The dynamics of the interplay between the domains and the other elements of the infrastructure of school mathematics is imagined and described (see page 23) 'A model of the essentials of mathematics P–10'.

Brief descriptions of the domains and some of their key concepts follow. They are illustrative of the key concepts required for a depth of understanding in each domain. The next stage of this review will provide further details of the key concepts in relation to the Levels of schooling described on page 1 of the CSFII and, as well, their relation to the learning outcomes in the CSFII. (See 'What might happen next?' page 23). The Levels of schooling are closely related to the stages learners move through as they develop their meanings in mathematics. This 'acscent of meaning' moves from 'informal mental processes and strategies based on [personal] meanings' to 'informal (child invented) algorithms' through 'computer or calculator assisted mathematics' to 'formal (taught) algorithms' and 'symbolic mathematics'. This ascent takes 'all the years of schooling' (Armstrong, 2004).

Number

Key concepts:

- Number as the size of a collection. Large and small numbers.
- Place value; zero; sequence and successor sequences.
- Prime numbers, factors and powers.
- Subsets of the natural numbers e.g. odd and even (as, for example, equivalence classes of modulus 2 arithmetic), those based on geometrical shapes (triangular, square, etc.) and the perfect numbers.
- Computations with numbers e.g. as aggregation (addition), dis-aggregation (subtraction), repeated aggregation of the same quantity (multiplication) and repeated dis-aggregation of the same quantity (division).
- Parts and wholes: units and fractions. Rational number and proportional reasoning.
- Negative numbers (e.g. historical views about natural number (Euler) can be used to introduce and develop a sequence of essential notions about negative numbers as data derived from everyday contexts such as money and change of temperature) …
- Words and symbols for number.
- Numbers in different bases.
- The number line as a metaphor for the real numbers …
- The golden ratio ϕ, and *pi*, π (for beauty and meaning and a simple introduction to the irrational and infinity).

Space

Key concepts:

- Insides, outsides and boundaries: curves (open and closed), shapes and surfaces.
- Enclosing space: two- and three-dimensional (e.g. the Platonic solids) objects.
- Position (e.g. maps, coordinates) and arrangement (e.g. tiling, including tilings of non-Euclidean worlds. For example, the sphere can be tiled with regular pentagons, by picturing it as a dodecahedron.)
- Symmetry and change – this relates to the central ideas of invariance and variance. E.g. self similar figures such as fractals; congruence and similarity.
- Other ways of imagining (or visually intuiting) space.
- Non-Euclidean geometry e.g. spherical geometry and mappings of the world.
- Topological space e.g. computer games such as Pacman and Asteroids. (These games are played in Euclidean worlds, but not in our standard Euclidean worlds: Pacman is played on a cylinder, and Asteroids is played on a torus.)
- Representations of spatial ideas: their transition from sketches of geometrical objects per se, to the concept of coordinates and representation of algebraic expressions as objects in the coordinate plane or space.

Measurement, data & chance

Measurement – number is the basis of all measurement (Euler)

Key concepts:

- Every measurement is a piece of data.
- Comparison of the qualities of objects.
- The need for units of measurement (co-measurability).
- Finiteness.
- Direct (e.g. counting) and indirect measurement.
- Estimation (reasonableness).
- Approximation and accuracy.
- Dimensional concepts (e.g. length, time and mass) and derived dimensional quantities (e.g. density).

Data – this domain exists across the curriculum.

The key concepts are *determinations* relating to:

- what question(s) are to addressed with which data
- whether it is possible to find or collect information appropriate to the question(s) and thus considered to be data. Appropriateness is assessed and evaluated according to the reliability and validity of the sources (e.g. the media and the Net) must be
- the accuracy of the data
- the linkage between the data and its interpretation each time the data is analysed, for instance, in terms of its status (sample or census) or the qualification of the connection between one quantity and another (e.g. causal or 'weak link').
- what to communicate to others by reporting what was done, what was found and what is considered OK and not so OK, and what might be done with it in the future.

Chance:

Key concepts:

- Randomness.
- The likelihood that events will occur in the short- and long-term.
- Measurement of chance as probability.
- Prediction.
- Simulation.

Structure

Key concepts:

- Patterns in number (e.g. the decimal forms of rational numbers; multiplicative models), geometry (e.g. the architecture of Federation Square is based on congruence and symmetry) and measurement (e.g. the use of place value as a model for comparing measurements).
- The identification of contexts for structure. For example: reading a map, understanding the argument of a newspaper article and the navigation of the web (same structure as following a street map; why are some websites easier to navigate than others?). And, in a more technical domain, graphs (trees and networks) making sense of mathematical systems such as arithmetic, equations e.g. polynomials, and sets of numbers and functions in terms of computability through to the structure of algorithms (even simple ones like personal and household procedures) and computer programs (in particular the programs that will be composed by the students in the course of their calculations).
- The development of an algorithm as a generalised way of dealing with specific process.
- Sets as collections: size, relationship between sets e.g. subsets, sets of sets (e.g. successive numbers, multiplication).
- Functions e.g. computable functions (partial and fully) such as the operations of arithmetic on pairs of natural numbers.
- Logic: order, axioms, formal proof.

Algebra. Considerations here include:

- distinctions between unknowns as place holders in equations and variables in expressions and functions, through to;
- the concepts of associativity, commutivity, and identity, and higher-order structures such as group, rings and fields.

Working mathematically

This area provides, if you like, the heartbeat of the school mathematics curriculum. It promotes the role of the central ideas in the other areas of school mathematics and highlights the connections between these areas. As well, it prompts considerations of how to implement the aims of school mathematics into the curriculum. It is a focus for study in its own right. It incorporates the material in the Reasoning and strategies strand in the CSFII and promotes the use of technology across the domains. This area provides a powerful reference point for teachers as they design and implement learning experiences for their students.

Working mathematically brings together the following aspects: Mathematical reasoning, Use of technology, Reference to the history of mathematics, and Contexts for the application of mathematical ideas and concepts.

Mathematical reasoning

Key concepts:

- A mathematical way of thinking (and questions that prompt mathematical thinking).
- Questions and problems (especially how to pose them and seek solutions to problems).
- Mathematical modeling.

Use of technology

Key concepts include the propositions that:

- the emphasis in using calculators and computer software in the curriculum should be on understanding and exemplifying the mathematics as well as doing computations
- calculators and computer software should be used to do repetitive calculations
- technologies of various kinds have been a part of mathematics throughout its history e.g. from the abacus to electronic calculators, and that in some instances the developments in mathematics have clearly preceded the related technology e.g. Boole's development of the algebra of logic on which modern computers are based
- more recent developments in technology such as computer algebra systems provide a comprehensive and coherent model of mathematics and mathematical structure.

Reference to the history of mathematics

Key concept:

- The comparison of mathematical concepts in western culture with those in other cultures e.g. time, measurement.

Key propositions included in this concept:

- Why was a particular mathematical idea, theorem or proof invented in a particular culture at a particular time, and how? For example, the approximation of circles by regular polygons leads naturally to the story of Archimedes and his systematic approximation to π.
- What mathematical ideas, theorems or proofs have been changed since they were first developed?

Contexts for the application of mathematical ideas and concepts

Key concepts:

- Cross-domain contexts e.g. the Pythagorean theorem as an equation relating areas of squares and a distance formula in Cartesian coordinates.
- The development of algorithms e.g. the development of recursive relations for Fibonacci numbers or programs to do repetitive operations (e.g. generate Pythagorean triples).
- Contexts for informal and formal proof. For example:
  Moving from the Fundamental Theorem of Arithmetic and the Euclidean algorithm needed to prove it, to Euclid's and others simple and beautiful proofs of the infinity of primes (e.g. proof by contradiction). From this it is easy and natural to discuss the sieve of Eratosthenes, the Great Internet Mersenne Prime project, and the twin-prime conjecture.

- Geometric proofs that ϕ and $\sqrt{2}$ are irrational numbers.
- Geometric proofs of Pythagoras can link to algebraic proofs of Pythagorean triples that lead naturally to the statement of Fermat's Last Theorem, and to a discussion of the fascinating history behind it. The $n = 4$ case (the only proof Fermat ever wrote down) is effectively just an infinite descent argument based upon the classification of Pythagorean triples.
- Proofs of straightforward propositions about numbers e.g. odd and even numbers.

Section 3: Future directions

A model of the infrastructure for the school mathematics curriculum P–10

Models can act as metaphors in that they provide a context for seeing, thinking about and understanding a concept. The role of the number line as a metaphor that relates number, measurement and geometry in mathematics is a case in point. The purpose of this section is to invite teachers to develop a model of the elements of the essentials of mathematics for P–10 as a metaphor that is real for them in their practice. In this context a metaphor can promote:

- creative problem solving (for example, we may 'see' differently something we have puzzled over for some time),
- metacognitive activity (as a cue to what kind of thinking should be done),
- reflections about the strengths and limitations of the metaphor (as one does with mathematical models), and
- the formulation of new versions of the metaphor or indeed of new metaphor (Perry, 1995)

In general, metaphors provide a guide for future action. Here is an example of one such model. The dynamics of the connections and interplay between the aims of school mathematics, the central ideas derived from the discipline, and the domains of school mathematics might be imagined as a geometrical model as follows:

> *Imagine a tetrahedron whose vertices at its base are the domains of number, space, and measurement, chance and data with structure at the apex. The vertices of this tetrahedron form, in turn, the vertices of the base of a pyramid whose apex is the 'working mathematically' domain. This pyramid is inscribed in a 'sphere' imagined to be the aims of school mathematics which are, of course, derived from the relationship between the discipline of mathematics and the societies and cultures in which the discipline exists. …This sphere might then be imagined to be a connected 'universe' comprised of other areas of the school curriculum and the disciplines in general.*

This model highlights the role of the 'Working mathematically' domain and its connections to the central ideas as starting points for the 'derived' (the sense that teachers make of the intended curriculum) and the 'planned' (that designed for a particular classroom) curricula.

What might happen next?

The next stage of this review of the essentials of mathematics P–10 might involve the communication of the infrastructure for the school mathematics curriculum described above to teachers, together with the production of materials that link the central ideas with the domains and their key concepts. These links might indicate, for example, which ideas are pre-requisite to others according to developmental levels of the students. These materials would also contain links to the learning outcomes in the CSFII. They would respond to the recommendations from the contributors to his review that depth is more important than breadth in the learning of mathematics. Fewer activities done in depth may be more valuable for learners than a superficial coverage of many topics over relatively short periods of time.

The criteria for seeking depth when developing materials for use in the classroom include:

- choosing mathematics which is significant in terms of the aims of school mathematics, the central ideas and their connections to the other domains
- allowing the activities in the materials to be open-ended in terms of where the learners may take them under the guidance of their teachers. Such guidance might involve spiralling back and forth through previously learned ideas and concepts. The time scale for these activities might be weeks, months or even years. In the case of activities involving the use of data sets these might be revisited and reviewed over the period of months or years across the domains of mathematics and other disciplines in the school curriculum.

The materials might be in two forms. First, materials which outline possible sequences for the concepts within the domains (and thus respond to the question ‘are there natural orders here?’). These materials would also outline possibilities across the domains in terms of the sequence of day-to-day and month-to-month programs in schools. Second, examples of rich activities that exemplify the connections and depth referred to above. Prototypes of these sorts of materials are available from various sources, for example, the VCAA publication ‘Reasoning and strategies Levels 1–6’ (Working Mathematically).

Final comments

The goodwill and idealism of the contributors was evident in their papers and their comments about the synthesis of their papers. While they are enthusiastic about the possibilities for developing the school mathematics curriculum that are contained in this report, they are aware that there is much to be done if its potential is to be realised in practice. The history of unrealised expectations associated with reform of the mathematics curriculum is ever present in their memories. The transition between the P–10 and the post-compulsory mathematics curriculum is often seen by teachers as a potential barrier to the implementation, in the P–10 curriculum, of ideals such as those expressed in this paper. The apparent need for continuity between the P–10 and the Victorian Certificate of Education (VCE) curriculum often overrides teachers’ dispositions to exercise their professional duty to develop P–10 curriculum that:

- meets the aims of school mathematics cited earlier, and
- includes open-ended activities that engage students in seeking depth in their understanding and experience of school mathematics.

Part of the next stage of this review will be to develop a strategy for the professional development of teachers so that they can understand and implement this contemporary view of the essentials of school mathematics. The contributors to this review strongly support such professional development – they know the importance of having teachers in classrooms who are confident and enthusiastic, and who appreciate and understand mathematics.

Section 4: References

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